Magic Spinners

NANAMIC Wednesday 6th July 2022

David Martin
david.martin@answers.me.uk

Magic Spinners

In this workshop we will learn how to generate magic squares of odd order and utilise these to create sets of intransitive spinners in which for a collection of three spinners A, B and C, if in some way A is better than B, and B is better than C, then it is not true that A is better than C. This builds on students' interest in magic squares, encouraging them to create their own, and develops their understanding of intransitive objects as has been experienced by many students in the game of Scissors, Paper and Stone where Scissors beat Paper, Paper beats Stone, but Scissors does not beat Stone.

You will benefit from pen and paper if you wish to complete activities.

Generating magic squares of odd order

As a 10-year-old, my interest in mathematics gained a significant boost when a neighbour gave me his Sunday School prize of the 1899 'The Boy's Own Annual'. I read with enthusiasm L. H. Hughes' article on how to construct a magic square of odd order, such as a 3 by 3 or 5 by 5 magic square.

Properties of a magic square

Each row, column and main diagonal sum to the same number, in these cases, 15 for the 3 by 3 magic square and 65 for the 5 by 5 magic square.

3 by 3 magic square

6	1	8
7	5	3
2	9	4

5 by 5 magic square

15	8	1	24	17
16	14	7	5	23
22	20	13	6	4
3	21	19	12	10
9	2	25	18	11

Generating magic squares of odd order

These, and other magic- squares of odd order, the article explained, can be completed using a simple rule.

Re-expressing this, first assume that the top edge is connected to the bottom edge and that the left edge is connected to the right edge so that as we fall off the left, we join round to the right and as we fall off the top, we join round to the bottom.

Place 1 in the top middle cell. Place the next number in the cell diagonally up and left, if that cell is empty, otherwise place it in the cell below, repeating this until all cells are filled.

5 by 5 magic square

15	8	1	24	17
16	14	7	5	23
22	20	13	6	4
3	21	19	12	10
9	2	25	18	11

Generating magic squares of odd order

We can illustrate this by entering the first 6 numbers in to a 5 by 5 magic square.

Enter 1 in the top middle cell. Moving up and left would take us off the top of the grid so we join round to the bottom row, one column to the left, where we place 2.

We move up and left to place 3. Moving up and left would take us off the left of the grid so we join round to the right column, one row up to place 4.

We move up and left from there to place 5. Moving on from 5, since the cell up and left is already filled, in this case by 1, we place 6 in the cell below 5.

Activity: Proceed to complete the magic square.

		1		
			5	
			6	4
3				
	2			

Generating magic squares of odd order

5 by 5 magic square

The top edge is connected to the bottom edge and that the left edge is connected to the right edge so that as we fall off the left, we join round to the right and as we fall off the top, we join round to the bottom.

15	8	1		
	14	7	5	
		13	6	4
3			12	10
9	2			11

Place 1 in the top middle cell. Place the next number in the cell diagonally up and left, if that cell is empty, otherwise place it in the cell below, repeating this until all cells are filled.

Generating magic squares of odd order

5 by 5 magic square

The top edge is connected to the bottom edge and that the left edge is connected to the right edge so that as we fall off the left, we join round to the right and as we fall off the top, we join round to the bottom.

15	8	1		
	14	7	5	
		13	6	4
3			12	10
9	2			11

15	8	1	24	17
16	14	7	5	23
22	20	13	6	4
3	21	19	12	10
9	2	25	18	11

Place 1 in the top middle cell. Place the next number in the cell diagonally up and left, if that cell is empty, otherwise place it in the cell below, repeating this until all cells are filled.

Zero sum magic squares of odd order

5 by 5 magic square

Something of the underlying symmetry of the magic square may be revealed by reducing each cell by the same amount, the value of the centre cell, so that the row and column sums are zero.

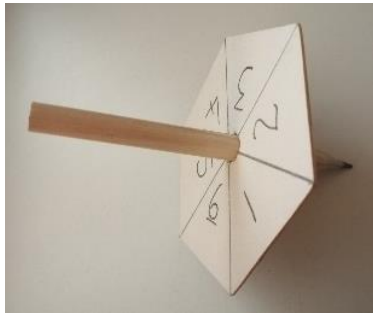
15	8	1	24	17
16	14	7	5	23
22	20	13	6	4
3	21	19	12	10
9	2	25	18	11

+2	-5	-12	+11	+4
+3	+1	-6	-8	+10
+9	+7	0	-7	-9
-10	+8	+6	-1	-3
-4	-11	+12	+5	-2

Rotate the 5 by 5 magic square through 180 degrees about the centre cell – each element has changed sign.

Dice and Spinners

Amongst my collection of dice, I have dice with 4, 6, 8, 12 and 20 numbered faces, each in the form of a Platonic Solid. My cubic dice have 6 faces, numbered from 1 to 6, though often these dice can have a corresponding number of dots, and can be manufactured to ensure that landing on each face is an equally likely outcome.

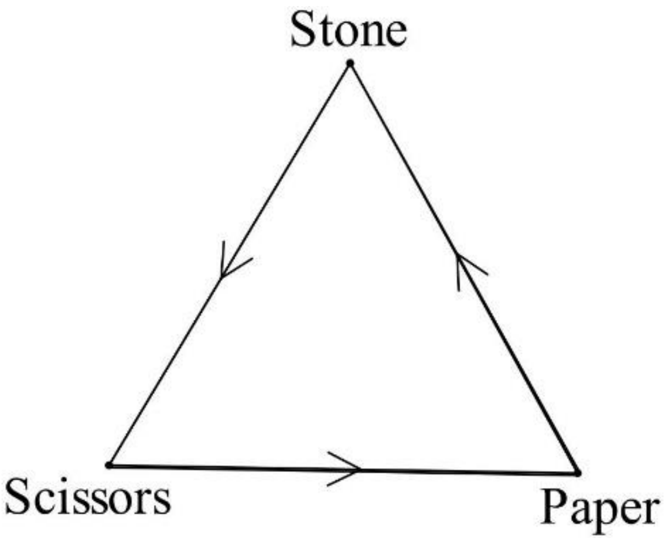


A spinner with a number of sections of equal size is an alternative to a dice for generating equally likely outcomes. Spinners can also have an odd number of equally likely outcomes, including 3 or 5, or can as in the case of dice be biased towards one outcome, e.g., by repeating an outcome.

The game of spinners

In the game of Scissors, Paper and Stone we have a set of three intransitive objects as scissors beat paper, paper beats stone, but scissors does not beat stone.

We shall express this as scissors > paper > stone > scissors and represent this in the triangle



In the Scissors, Paper and Stone game Scissors will always beat Paper, Paper will always beat Stone and Stone will always beat Scissors.
We will now introduce a game where certainty of winning is replaced by a probability of winning.

The game of spinners

Leo and Mike play a game with several rounds. Each round Leo and Mike will each spin a spinner and the one with the higher score will win two points. If they have the same score each will be awarded one point.

If Leo and Mike play with the same spinner, then we might expect that after several rounds Leo and Mike will have about the same total number of points.

If Leo and Mike select in turn from a set of three spinners S1 , S2 and S3, where

S1 has three equally likely outcomes of 1, 6 and 8

S2 has three equally likely outcomes of 3, 5 and 7

S3 has three equally likely outcomes of 2, 4 and 9

The average score for each spinner is the same, 5. If Leo chooses spinner S1, does it matter which spinner, S2 or S3, Mike chooses?

The game of spinners

Let’s draw up a grid and begin by examining the 9 equally likely pairs of scores on the two spinners S1 and S2, noting which spinner has the higher score in each case e.g., if spinner S1 scores 6 and S2 scores 5 then S1 wins. From the completed table we note that S1 wins over S2 with an advantage of 5:4 i.e., with probability 5/9. That is, we would expect S1 to win over S2 after several rounds.

S1\ S2 scores	3	5	7
1	S2	S2	S2
6	S1	S1	S2
8	S1	S1	S1

Writing this as S1 › S2, we also find that S2 wins over S3 with an advantage of 5:4 i.e., S2 › S3

So, if S1 wins over S2 and S2 wins over S3, we might reasonably expect that S1 wins over S3.

i.e., S1 › S2 › S3, implies S1 › S3?

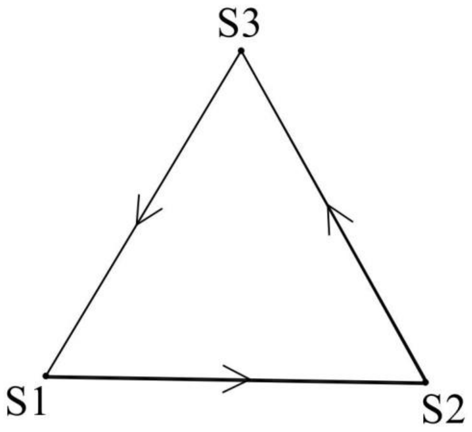
The game of spinners

But repeating the analysis for S1 and S3, we find that the reverse is true, and that S3 > S1

S1 > S2 > S3 > S1

As although S1 > S2 and S2 > S3 we do not have S1 > S3, we have three intransitive spinners S1, S2 and S3.

We can illustrate this in the triangle.



So, if Leo chooses spinner S1, then Mike will have an advantage choosing S3

The game of spinners

As an alternative approach to comparing spinners we can draw up a table that shows the order in which the digits 1 to 9 have been allocated to S1, S2 and S3. Comparing the S1 row against the S2 row, we see that the 1 does not beat any in the S2 row while the 6 beats the 3 and 5 in the S2 row and the 8 beats the 3, 5 and 7 in the S2 row; that is S1 beats S2 five times out of the total number of 9 paired spinner outcomes.

S1	1					6		8	
S2			3		5		7		
S3		2		4					9

Activity: Convince yourself that S2 > S3

The game of spinners

Now, returning to the 3 by 3 magic square, in particular to its rows, the rows 1, 2 and 3 of the 3 x 3 magic square have been used to create the spinners S1, S2 and S3

I have a collection of cubic dice with blank faces, on which can be written six equally likely outcomes

If we wrote on three cubic dice D1, D2, D3 the following equally likely outcomes, by repeating each spinner outcome.

D1 1, 1, 6, 6, 8 and 8

D2 3, 3, 5, 5, 7 and 7

D3 2, 2, 4, 4, 9 and 9

We would obtain three intransitive dice with

D1 > D2 > D3 > D1

The game of spinners

If we had used the columns A, B and C of the 3 x 3 magic square to create the spinners SA, SB and SC, where

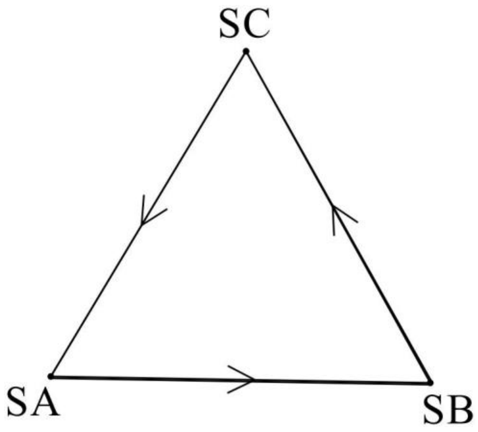
Spinner SA has three equally likely outcomes of 2, 6 and 7

Spinner SB has three equally likely outcomes of 1, 5 and 9

Spinner SC has three equally likely outcomes of 3, 4 and 8

SA > SB > SC > SA. We have three more intransitive spinners SA, SB and SC

We can illustrate this in the triangle.



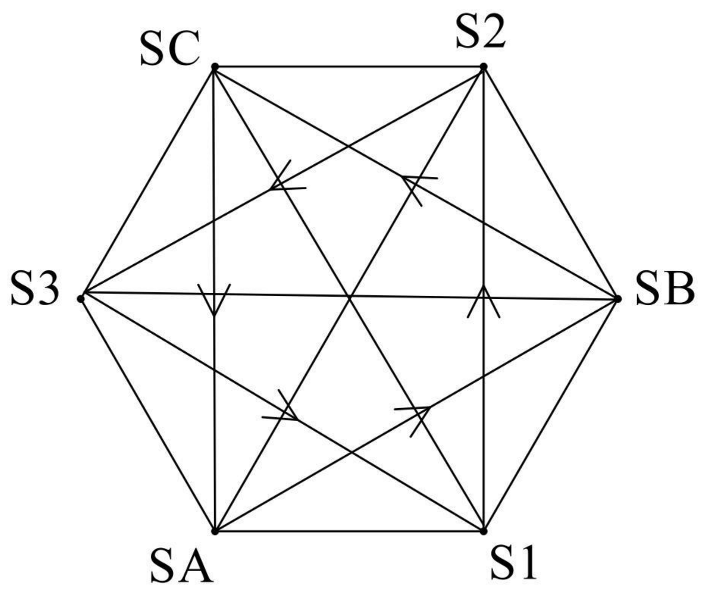
The game of spinners

If we mix the two sets of spinners, we note that after Leo has selected a spinner from one of the sets, there would be no advantage for Mike choosing one of the spinners from the other set

We will express this as SA = S1, SA = S2, SA = S3, ….

and write (S1, S2, S3) = (SA, SB, SC) as a shorthand for this.

We can illustrate this in a hexagon; where a line with an arrow represents a 5:4 advantage, and a line without an arrow represents no advantage.



The game of spinners

We can extend this analysis to the 5 by 5 magic square.

If we create from the rows of the 5 by 5 magic square a new set of five spinners, each with five equally likely outcomes of

S1 1, 8, 15, 17, 24

S2 5, 7, 14, 16, 23

S3 4, 6, 13, 20, 22

S4 3, 10, 12, 19, 21

S5 2, 9, 11, 18, 25

Then S1 > S2 > S3 > S4 > S5 > S1 with a 14:11 advantage

So, if Leo chose S2, Mike could choose S1 to have an expected advantage.

In addition, S1 > S3 > S5 > S2 > S4 > S1 with a 13:12 advantage

The game of spinners

If we create from the columns of the 5 by 5 magic square a set of five spinners, each with five equally likely outcomes of

SA 3, 9, 15, 16, 22

SB 2, 8, 14, 20, 21

SC 1, 7, 13, 19, 25

SD 5, 6, 12, 18, 24

SE 4, 10, 11, 17, 23 We find that

SA > SB > SC > SD > SE > SA with a 14:11 advantage, and

SA > SC> SE > SB > SD > SA with a 13:12 advantage

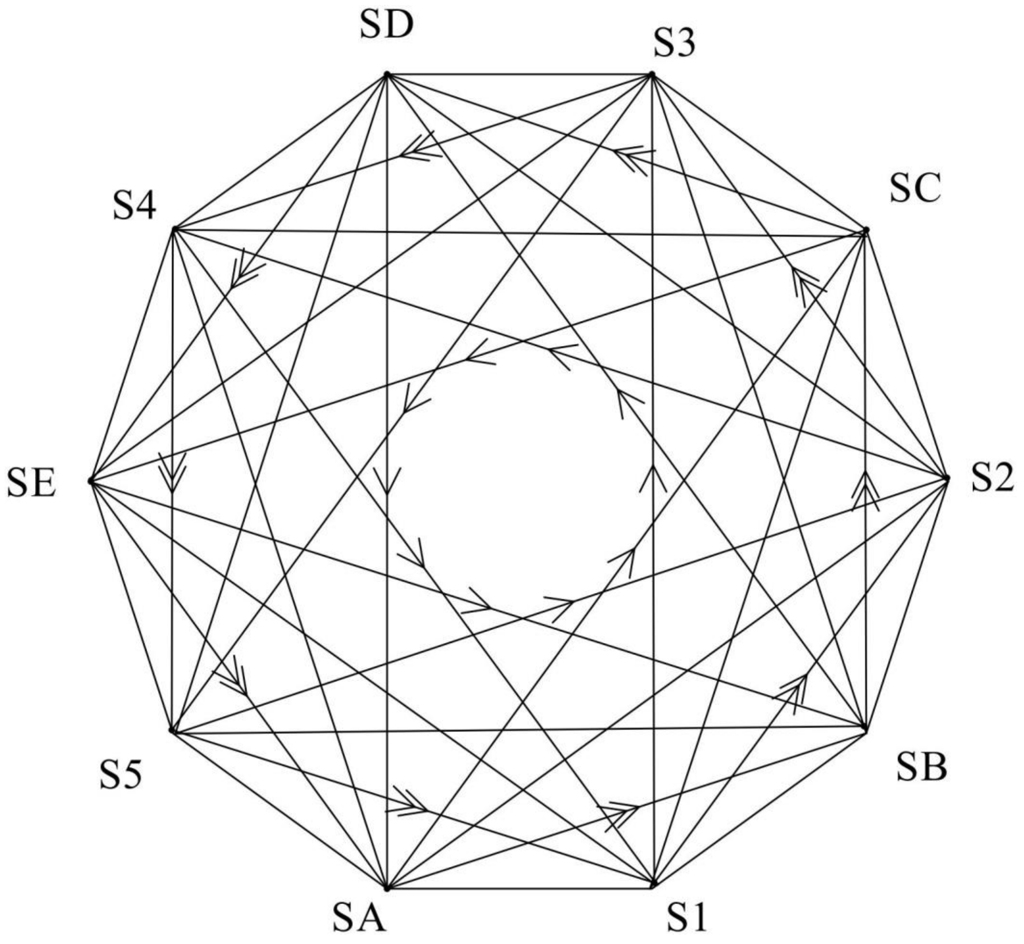
So, if Leo chose SA, Mike could choose SE to have the maximum expected 14:11 advantage.

The game of spinners

(S1, S2, S3, S4, S5) = (SA, SB, SC, SD, SE)

That is, if Leo chose from one of the sets of 5 spinners there would be no advantage for Mike if he chose a spinner from the other set of 5 spinners.

This can be illustrated in a decagon; where a double arrow line represents a 14:11 advantage, a single arrow line represents a 13:12 advantage, and a line without an arrow represents no advantage.



The decagon can be used to find sets of 3 intransitive spinners such as S1 > S2 > S4 > S1

Icosahedral dice

Extending the approach of two copies of each of the 3 outcomes of the S3 spinners to form cubic dice, four copies of each of the 5 outcomes of the S5 spinners can be used to form 20-sided icosahedral dice.

Conclusion

We have shown how to create a magic square of odd order and how either the rows or columns of these can, surprisingly, be used to create sets of intransitive spinners. You may wish to complete the analysis for a higher order magic square. There is much to investigate and to reason why, including progress towards a proof or otherwise that this will always be true.

Thank you

An article based on this material has been accepted for publication in the MA's Mathematics in Schools.

David Martin (david.martin@answers.me.uk)