Principles of problem solving

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Principles of problem solving

In this session - we will take a look at principles of problem solving, examining different approaches through the lens of four or five problems that we will be solving together.

Problem solving – solving a problem for which finding a suitable approach is part of the problem.

We wish to develop skills in mathematical

- Fluency
- Resilience
- Problem solving
- Examinations
- Communication
- Computing
- Comprehension
- Collaboration
- Techniques
- History…

Principles of problem solving

1. Understand the problem e.g.,

- Restate the problem using your own diagram and words
- Is there enough information to enable you to find a solution?

2. Devise a plan e.g.,

- Look for a pattern
- Solve a simpler problem

What other approaches could you list in Devise a Plan?

3. Carry out the plan e.g.,

- Persist with the plan
- If it continues not to work discard it and choose another.

4. Look back e.g.,

- Could you have solved it in a different way?
- Could you use the method for other problems?

George Pólya's four principles (https://en.wikipedia.org/wiki/How_to_Solve_It) - cyclic

Devise a plan – some approaches:

1. Guess and check
2. Eliminate possibilities
3. Use symmetry
4. Consider special cases
5. Use direct reasoning
6. Solve an equation
7. Find a counterexample
8. Estimate expected results
9. Look for a pattern
10. Draw a picture
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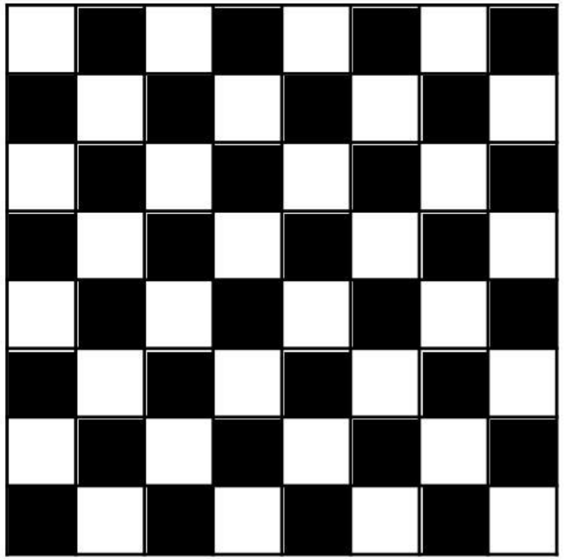
Devising a plan – A chessboard problem



There are 64 ‘unit’ squares on a chessboard. And the chessboard itself is an 8 x 8 square.

There are squares of intermediate sizes on the board.

1. How many squares are there in total?

2. How many squares would there be on an n x n board?

What would be your plan?

Solve a simpler problem

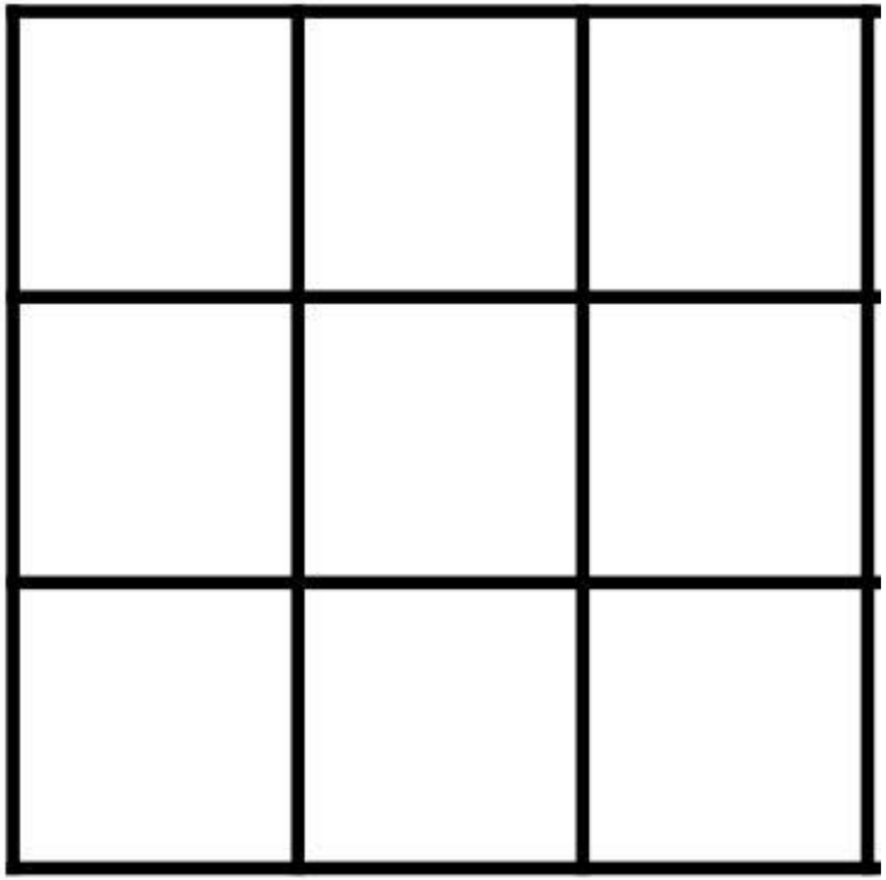
Draw a picture

Make an orderly list

Look for a pattern

Use a formula

Solving a simpler problem
How many squares do you see?

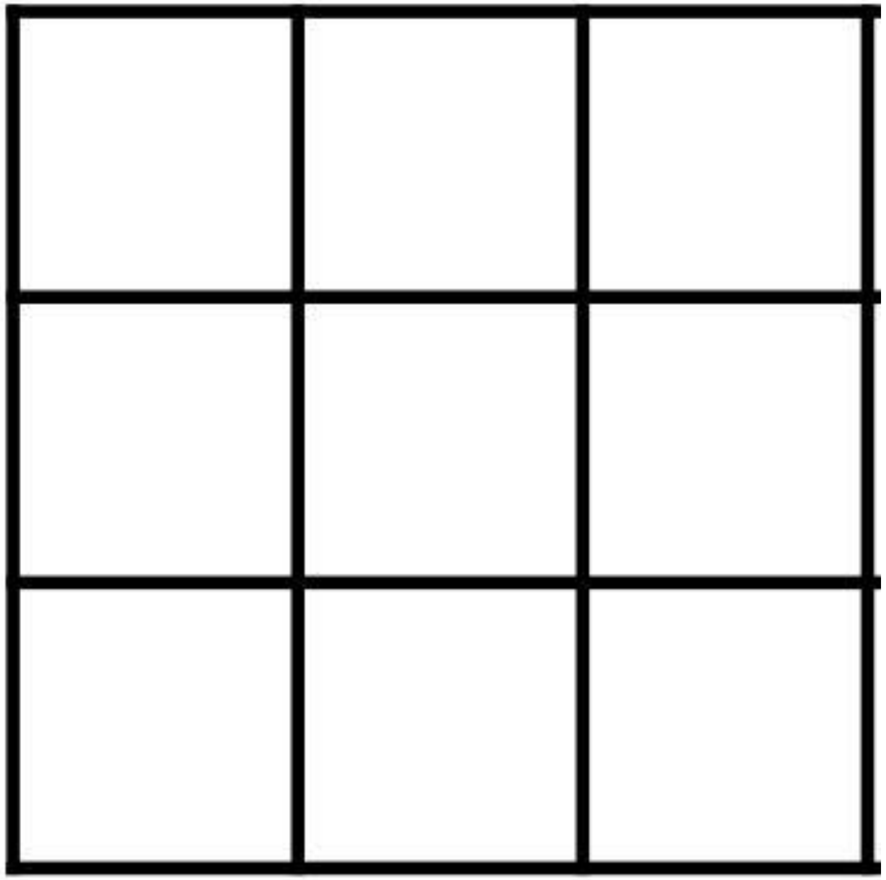


Solving a simpler problem
How many squares do you see?

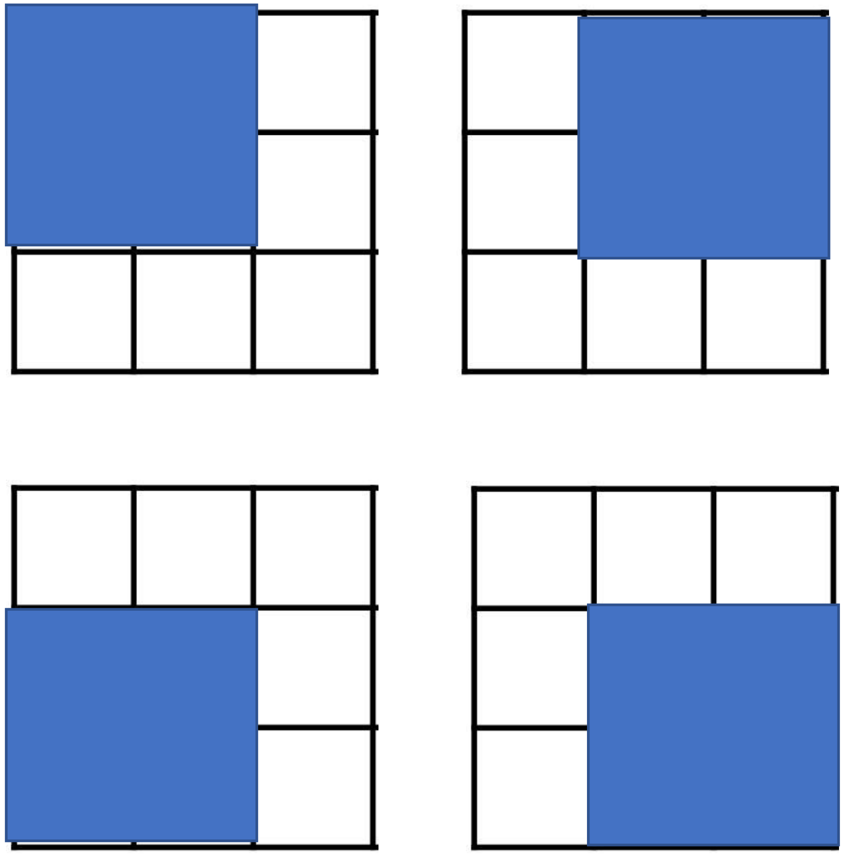
Poll 1

How many squares do you see? [Select one]

1. 9
2. 13
3. 14
4. None of the above



How many squares? - Looking for a pattern



Size	1	2	3
Number	3 x 3 = 9	2 x 2 = 4	1 x 1 = 1

A total of $1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$

In general, using the formula

$1^2+2^2+3^2+\ldots n^2 = n(n+1)(2n+ 1)/6$

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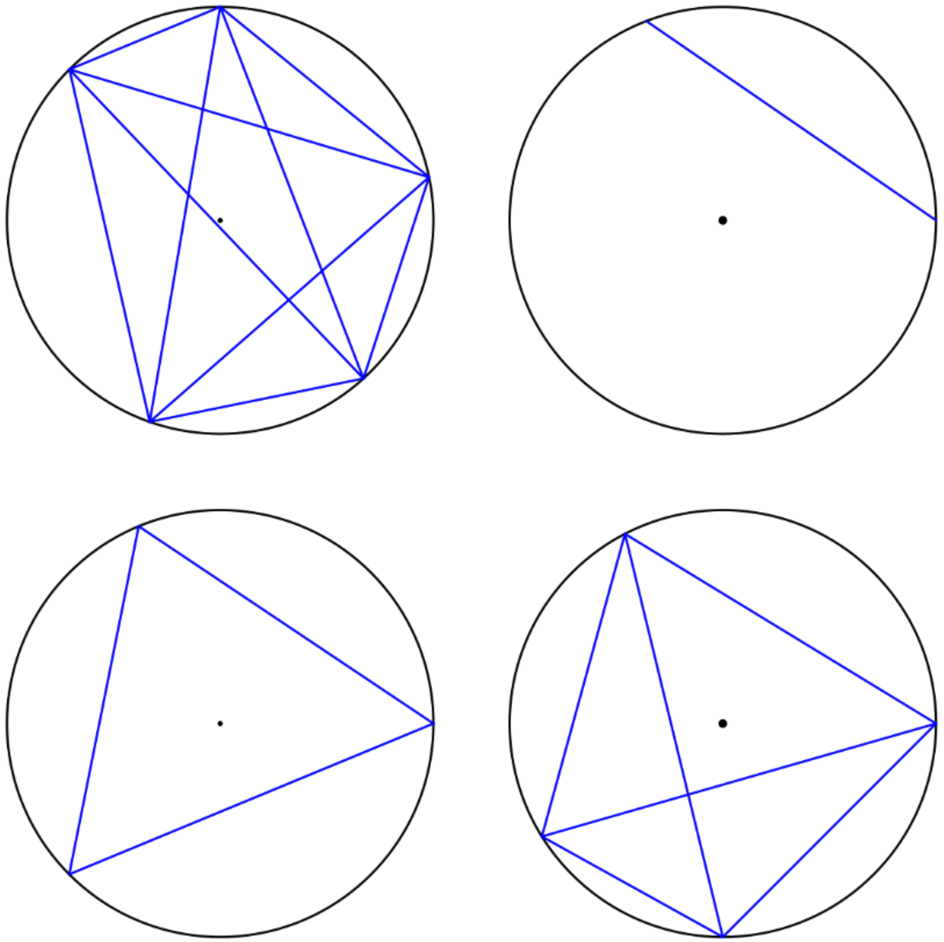
Pattern spotting is just the start - Circle regions

N points are positioned on a circle and connected by chords in all possible ways so that no three chords cross at the same point inside the circle. Into how many regions will the chords partition the circle? The diagram illustrates this for 2, 3, 4 and 5 points.



Circle regions – extending the pattern

Formula 2^{N-1}

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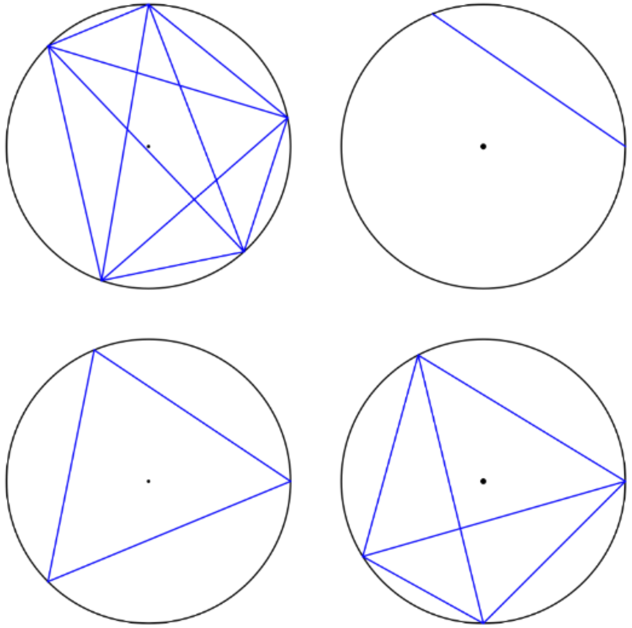

N points	Regions
2	2
3	4
4	8
5	16
6	32
7	64
8	128
9	256
10	512

Circle regions – finding a counterexample

Formula $1 + {}_{N}C_{2} + {}_{N}C_{4}$

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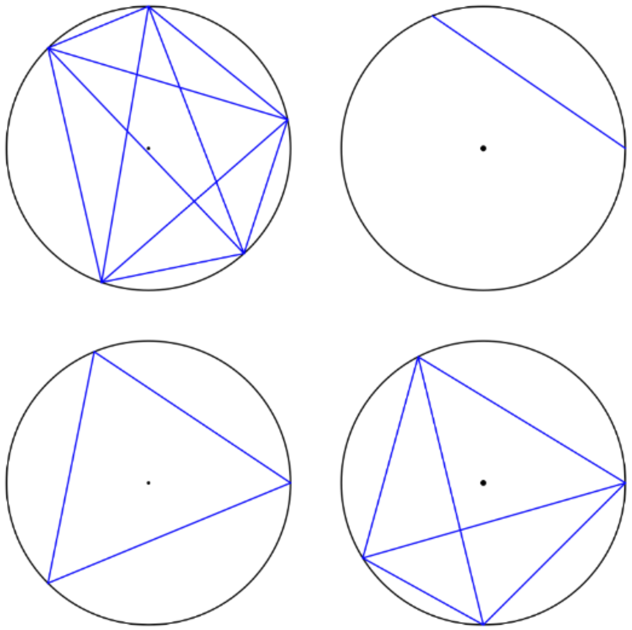

N points	Regions	Actual
2	2	2
3	4	4
4	8	8
5	16	16
6	32	31
7	64	57
8	128	99
9	256	163
10	512	256

Making connections to Pascal’s triangle Highlighted Nth row $1 + {}_NC_2 + {}_NC_4$

								1								
							1		1							
						1		2		1						
					1		3		3		1					
				1		4		6		4		1				
			1		5		10		10		5		1			
		1		6		15		20		15		6		1		
	1		7		21		35		35		21		7		1	
1		8		28		56		70		56		28		8		1

Row	Sum
0	-
1	-
2	-
3	-
4	8
5	16
6	31
7	57
8	99

Devise a plan – some approaches:

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Matching pages problem

A magazine has 92 pages and consists of a number of A3 sheets (each containing 4 pages) stapled together.

What page must be printed alongside page 12 to achieve correct page numbering?

Matching pages solution

What page must be printed alongside page 12 to achieve correct page numbering of a 92 page magazine consisting of 4 page sheets?

As the front page needs to be printed alongside the back page then pages 1 and 92 must be printed alongside each other.
Moving onto page 2 this will need to be printed alongside page 91, and page 3 alongside page 90, and so on until page 12 is alongside page 81

Looking at this another way, the sum of each pair of adjacent pages is a constant (an invariant) of 93, hence page 12 must be printed adjacent to page 93 – 12 = 81

Example of another invariant: For simple 3D polyhedra the sum $V – E + F = 2$
e.g. for a cube $V = 8$, $E = 12$ and $F = 6$ and $8 - 12 + 6 = 2$

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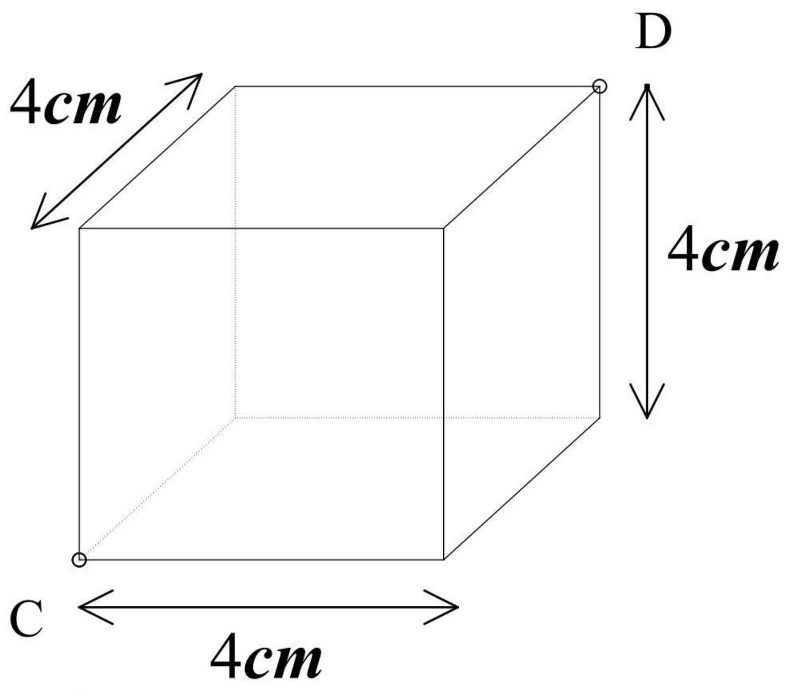
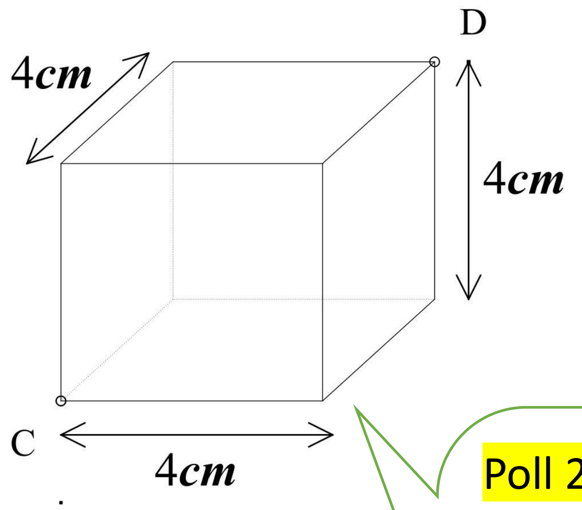
Drawing a picture - Shortest path problem



An insect is at a bottom corner C of an open topped cubic box with sides of 4cm and wishes to crawl to the diagonally opposite top corner D. What is the length of the shortest path that it could take?

Drawing a picture - Shortest path problem



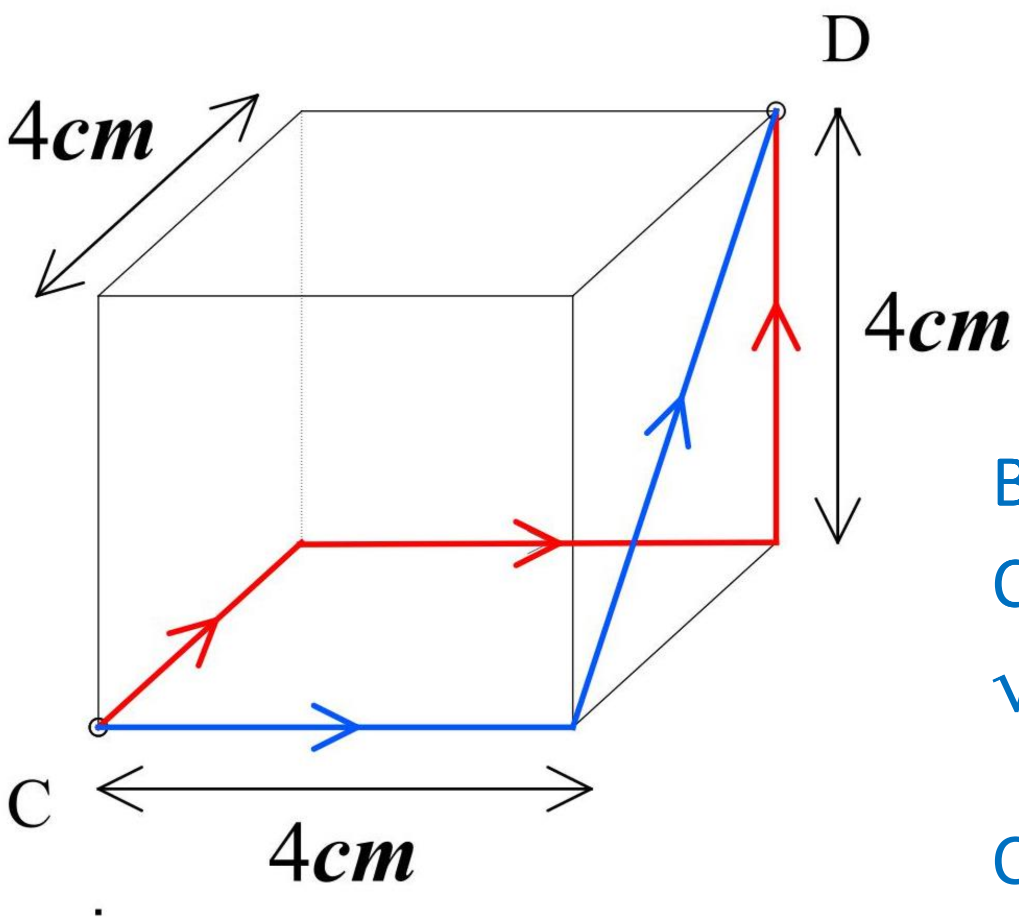
An insect is at a bottom corner C of an open topped cubic box with sides of 4cm and wishes to crawl to the diagonally opposite top corner D. What is the length of the shortest path that it could take?

Poll 2

Which of the following is closest to your answer to the shortest path length? [Select one]

1. 6.93 cm [CD direct through air]
2. 8.94 cm [CD shortest, found by net]
3. 9.66 cm [CAD one diagonal]
4. 12 cm [CABD along the edges]

Red and Blue paths

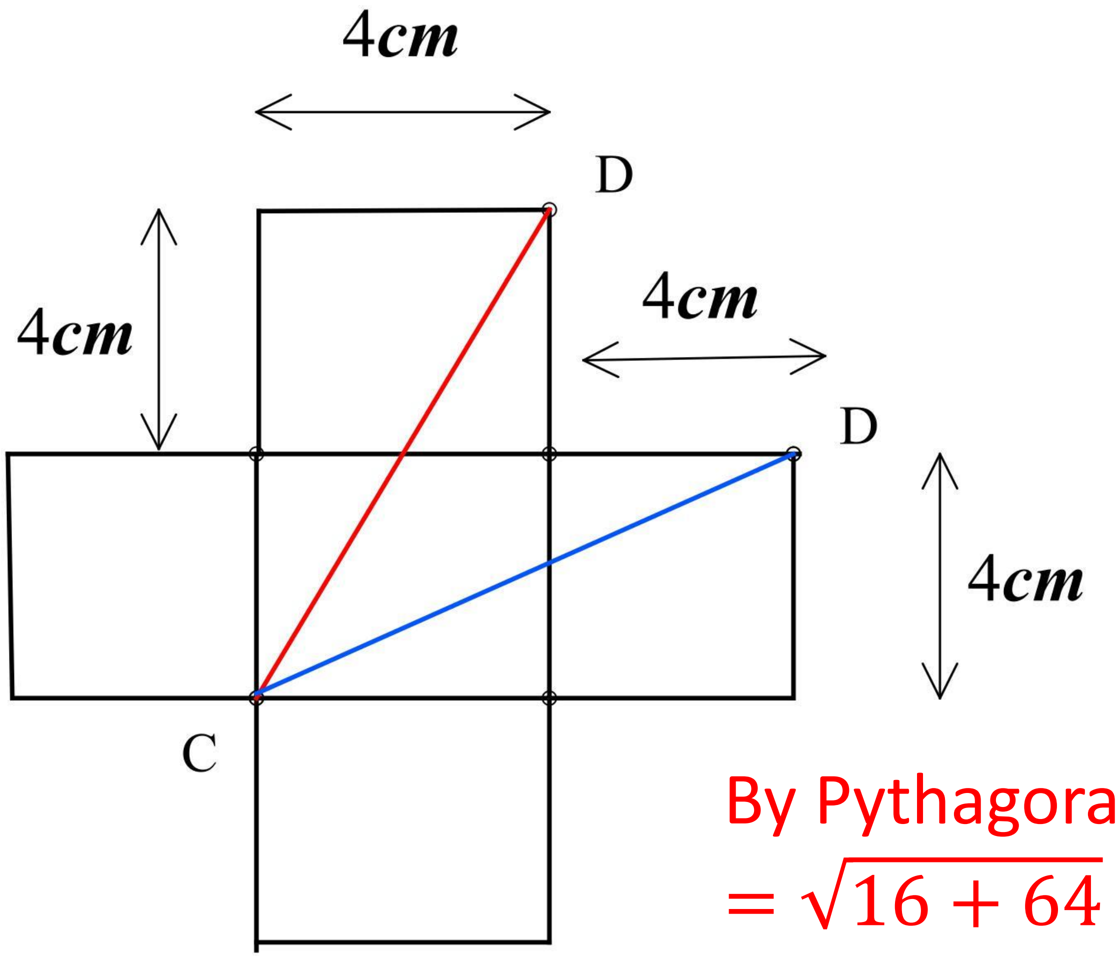


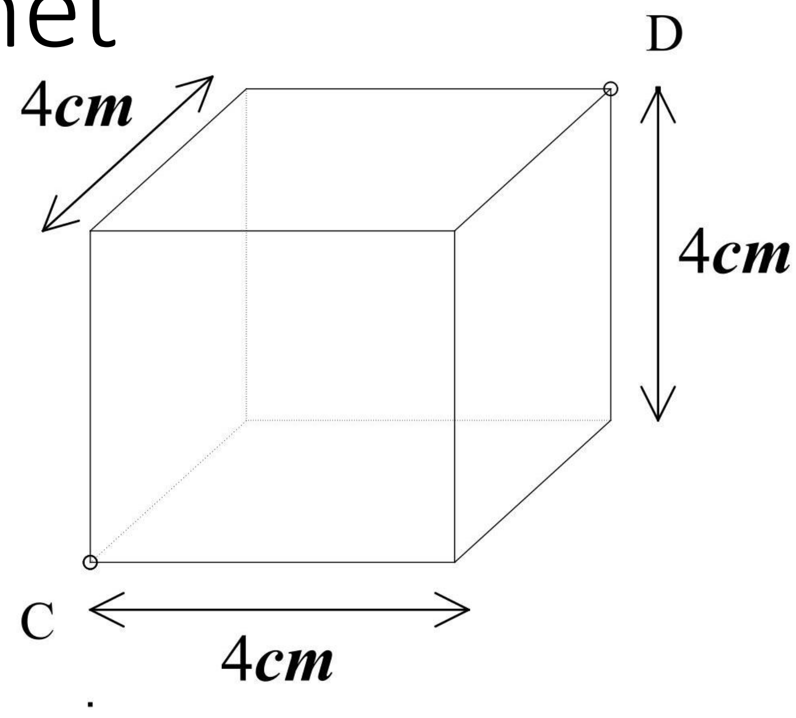
C to D distance = 4 + 4 + 4 = 12 cm

By Pythagoras the upwards leg of the C to D distance = $\sqrt{4^2 + 4^2} = \sqrt{16 + 16} = \sqrt{32} = 4\sqrt{2} = 5.66$

C to D distance = 4 + 5.66 = 9.66 cm

Shortest path – drawing the net





By Pythagoras the C to D distance = $\sqrt{4^2 + 8^2}$
$= \sqrt{16 + 64} = \sqrt{80} = 4\sqrt{5} = 8.94\ cm$

Compare with 1 side and 1 diagonal distance of 9.66 cm

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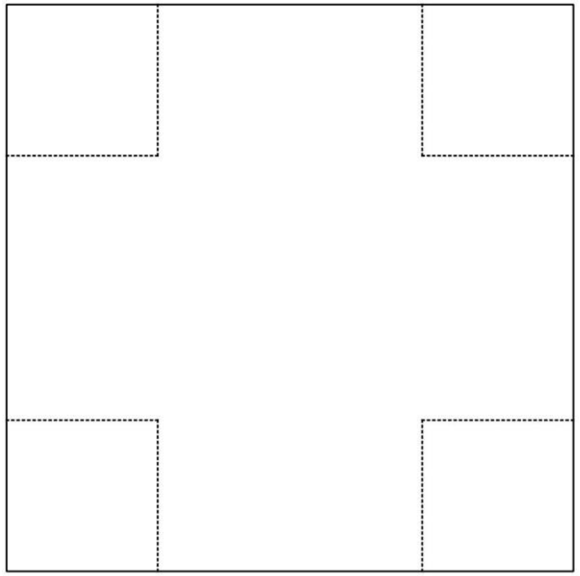
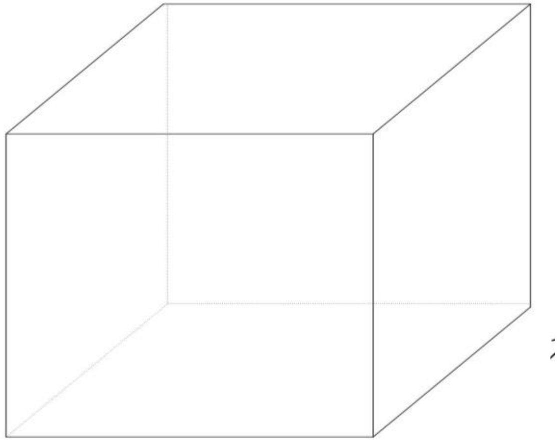
MaxBox problem

Fiona has a 30cm x 30cm square sheet of metal and needs to construct an open box (i.e., with no top) with a square base. She intends to cut out four corners and fold up the resulting four tabs to make the four sides. What is the maximum volume of box that she can construct in this way?

(If she cuts out four 3cm x 3cm corners, the base of the box will be 24cm x 24cm and its height will be 3cm, alternatively, cutting out four 6cm x 6cm corners will produce a box with dimensions 18cm x 18cm x 6cm.)

Leaving for the moment the details of the solution - what would be your plan to solve this problem?





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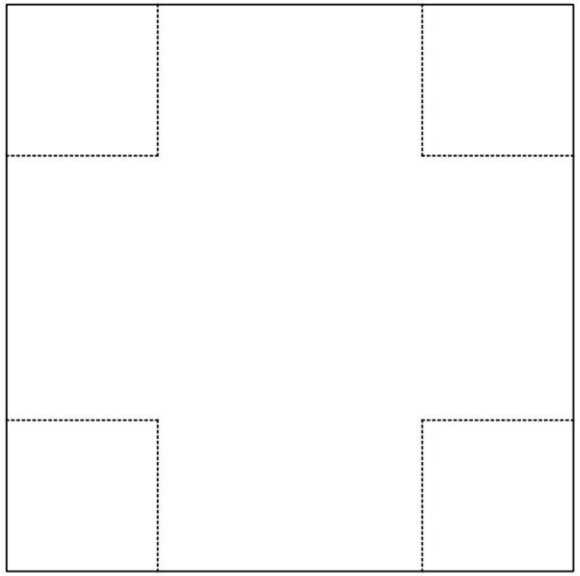
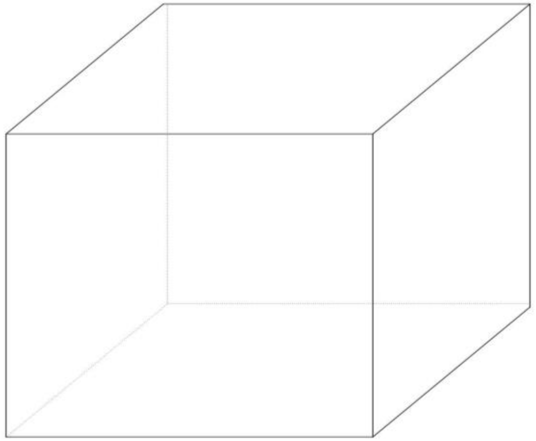


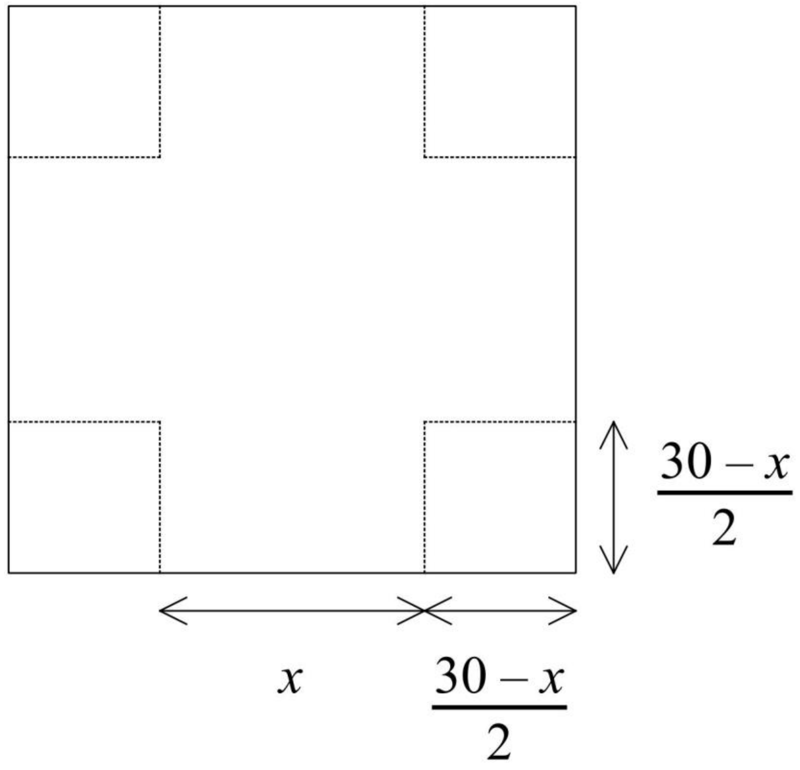


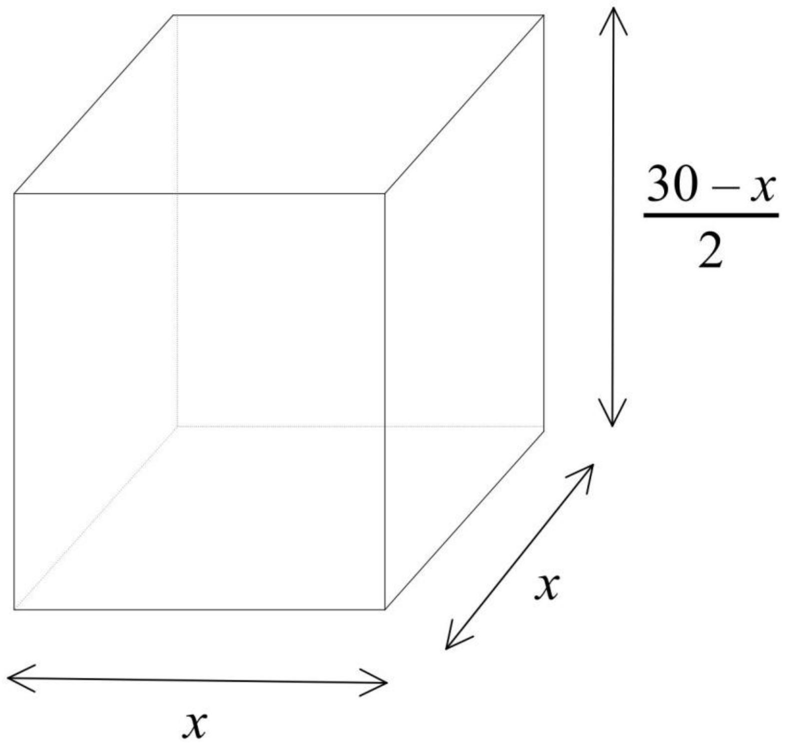
Poll 3

Which of the following, if any, would you use to solve the MaxBox problem? [Select one or more]

1. Table
2. Graph
3. Calculus
4. None of the above

MaxBox net and box

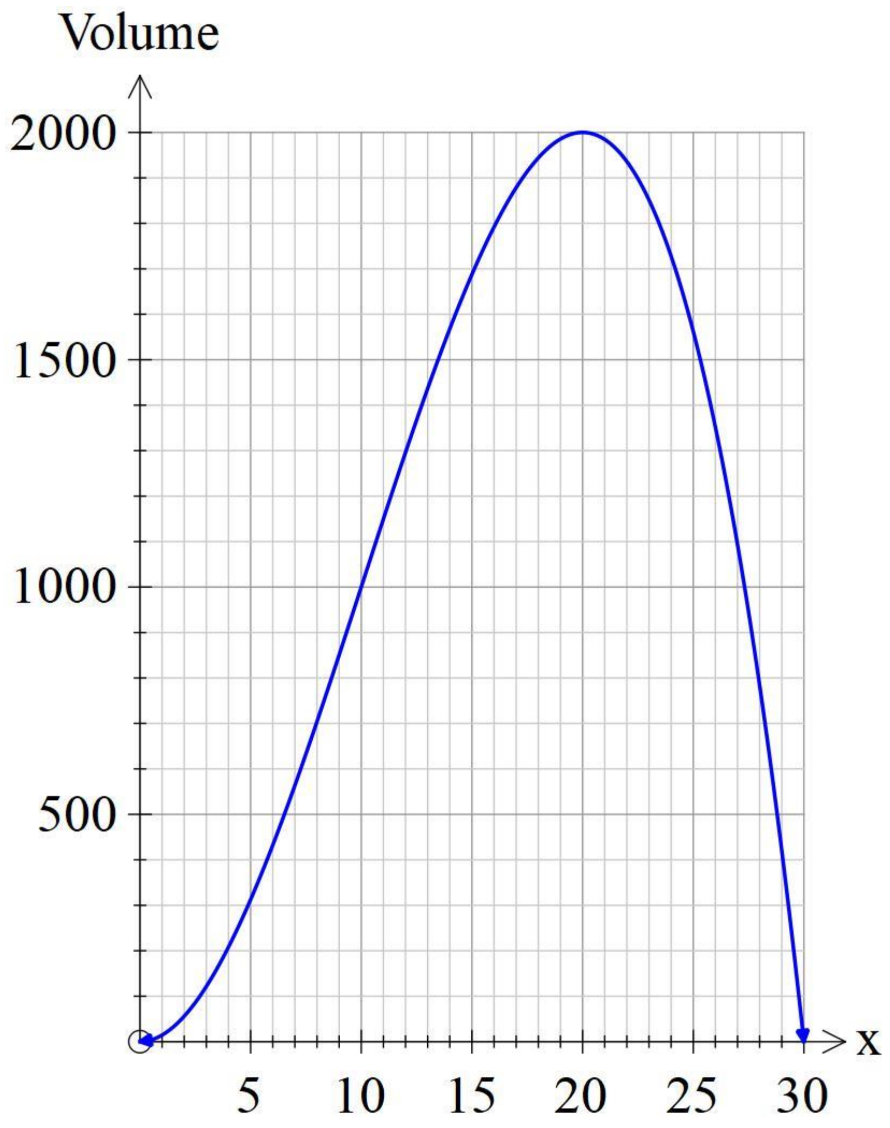




With square base of x cm

x	h	$V=x^2h$
0	15	0
2	14	56
4	13	208
6	12	432
8	11	704
10	10	1000
12	9	1296
14	8	1568
16	7	1792
18	6	1944
20	5	2000
22	4	1936
24	3	1728
26	2	1352
28	1	784
30	0	0

Three sample approaches - using a table, graph, calculus



$$Volume\ V = \frac{x^2(30-x)}{2}\ cm^3$$

$$V = 15x^2 - \frac{x^3}{2}$$

$$\frac{dV}{dx} = 30x - \frac{3x^2}{2} = 0$$

$$x\left(30 - \frac{3x}{2}\right) = 0$$

$$x = 0\ or\ x = 20$$

$$\frac{d^2V}{dx^2} = 30 - 3x$$

<0 when x=20, hence maximum here

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What might we value in addition to getting a solution?

1. Collaborative work in problem solving
2. Sustained work on problems and resilience
3. Using computer software to solve problems
4. Planning on how a problem might be solved
5. Assessment, improvement and completion of another’s method
6. Appropriate confidence in answers
7. Comparison of methods, why one method rather than another
8. Ability to explain a problem and its solution to another
9. The creation of notation to help solve a problem
10. Understanding the application of mathematics

Thank you

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