


# Informal Proceedings

# Editorial

Sue Pope

*BCME9 Programme chair*

It has been a great pleasure to edit the BCME9 Informal Proceedings. They give an indication of the remarkably rich programme available to participants 3 to 6 April 2018. The refereed papers are being edited by members of BSRLM and will be available later in the year.

Sincere thanks to all the contributors.

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# The New Jericho: Why we must break down the walls of the mathematics classroom

Graeme Austin

**Keywords: mathematics in context; real world; real-life**

## Why is mathematics boring?

While the vast majority of those who teach mathematics in secondary schools enjoy the subject and find it interesting, the same cannot be said for the pupils sitting in the same classroom. There is a fundamental disconnect between the way this academic subject is taught and almost every other academic discipline.

In geography, when pupils encounter a discussion about rivers or towns, they understand immediately what is being discussed. If there is a topic on earthquakes, those in the UK and many other parts of the world may have no first-hand experience, but a video is shown to give an instant context in which to situate the discourse. A tune is played in music so that a baroque style is available to anyone in the room to hear.

Even though people outside the classroom use various aspects of mathematics in their day-to-day lives, pupils are shielded from this simple truth by the examples chosen by teachers to explain the latest topic under discussion. These examples tend to be stylised, i.e. they are simplified to the point where just the specific current idea is exposed. This means we divorce pupils from the context of the problem as it might be in normal life.

Pupils ask me: what is the point of learning about simultaneous equations? I explain I used them when I ran my own business to make judgements as to when to raise the price of my products, using past sales data. I turned equations into cash and kept the teenagers' attention.

## Inside versus outside polarity

Schools exist in isolation from the rest of the world partly because of the physical nature of the institutions: typically, they are surrounding by a wall and there is a gatekeeping system before you can enter the premises. These facilities exist for good reason, but their very physicality creates a misconception in the adults who work within those walls. The school may appear separate from the world, when in fact it is as much a part of the so-called real world as anything else on this planet. One hears teachers and educational leaders describe the 'real world' as something that exists beyond the school gates, but they are wrong – as every pupil knows.

Every time an adult in school repeats this falsehood, they reinforce the idea there is some fundamental disconnect between a mathematics classroom and the rest of planet Earth. This enables teachers to believe that, by simplifying problems into something unrecognisable, they are helping their pupils learn. Instead, they suck the context out of the example and prevent pupils from building resilience in problem-solving scenarios.

## Stylised Examples

When I was training, I came across a concept which has stuck with me: *MathsWorld*. This is a mythical place in which most examples inhabit. People have names that are alphabetical (Anne, Bob, Chuck…) and they behave in certain pre-defined ways: everyone acts as though the world they are living in is rational. The probability of me taking a purple ball out of a bag, which contains three red and four purple, is zero. In *MathsWorld*, it is four sevenths. And we teach pupils that the latter answer is 'correct'.

I used to think I was alone in finding *MathsWorld* so strange and one day I pointed at a question I'd carefully placed on the board and laughed. It was classic *MathsWorld* fodder. Having explained my amusement to the assembled pupils, they agreed with me and volunteered to produce a list of dull *MathsWorld* scenarios, as shown in figure 1.

Cliche questions

- Boys and girls
- Cats and dogs
- Apples and oranges (fruits)
- Birthdays and months
- Exams
- Money
- Lunches
- Marbles
- Getting to school
- Balls in a bag
- Cars
- Sweets
- Pencils and stationary
- Buying things like book

Figure 1: MathsWorld questions compiled by Year 8 pupils

## Is there any real-world mathematics out there?

A movement that has received a recent boost in the UK is that of Realistic Mathematics Education (RME). Devised in the Netherlands, the aims are to be less prescriptive about which methods are used to solve a problem and to ground the problems themselves in something that pupils can relate to – hence the term 'realistic'.

While the idea of introducing greater connection between pupils and problems is to be lauded, I do not believe the *laissez faire* approach to methods is one I can support. Some ways of doing things are more efficient than others – and some contain a simplicity that leaves you in awe of the thought process which generated it. Mathematics is poetry and the journey to the solution matters. A lot. As much as I would like to support the initiative, I find myself unable to do so.

In my quest for useful resources that are borne from the real world, I thought I would spend a happy few hours in the National Curriculum topic of real-life graphs. The phrase itself drips with expectation of a world I know well: the real one.

Imagine my disappointment when I discovered this was merely a lazy phrase to describe a topic which looks at compound measures and interpreting graphs of more stylised examples! There is a GCSE question where the driver of a motorised vehicle travels at 56 mph for 40 seconds before slamming to a halt on a pointless journey to nowhere. I almost fell into my coffee mug. 'Disappointed' was redefined for me the day I found myself covering that example with some year 9 pupils.

## Is there any real mathematics out there?

Mathematics in the context of what happens in the world is harder to find for a teacher than you might think. There are a handful of websites that contain problems derived from direct experience of living in the world. These are primarily STEM-based and tend to live within the Applied Mathematics category.

As for Pure Mathematics work, like algebra or trigonometry, there is a dearth of published material. To counter this problem, I decided to set up a website to enable teachers to share their resources that do contain real world examples – at www.realworldmaths.education – and the response since the soft launch in January 2018 has been positive.

It is a free repository for good quality mathematics examples: no more and no less. The primary objective is to help me find better examples and I believe other teachers hope to achieve the same thing.

Why am I bothering? Because pupils engage most when they see a connection between what they are doing in the classroom and what they do outside that narrow environment. I have seen this work when I introduce Pythagoras using flat-pack furniture. The challenge is to find sufficient problems to cover the entire curriculum.

## Where next?

First, I hope to grow the pool of examples I have to work with and build the website, so others can gain from the efforts of others – in a sharing and caring manner without cost to the hard-pressed teacher. This is the way other resource sharing sites work. My aim is to ensure there are no fees levied for downloading a resource.

Second, and I acknowledge this may never happen, I would love to have a book of real-world resources to use but that aspiration is some way off: the initial need is to build up resources for use in the classroom. Of course, there are many side benefits to deploying real-world examples: they constitute the best form of problem-solving questions and prepare pupils for the forms of questioning now contained in the new GCSEs and A-levels. It boosts their resilience and helps them progress faster.

Third, I live in the world, I teach in the world, my pupils exist in the world and so does my school. I see no reason why the problems I set in my classroom should be any less relevant to my pupils as the questions they are asked in geography.

# Reasons to Reason in Primary Mathematics and Science

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**Keywords: reasoning; thinking; mathematics; science; primary**

## Is reasoning the currency of the future?

Much is said in educational circles about the importance of learners being able to reason. In the information age our need to remember and recall huge quantities of knowledge may be diminishing. At the same time the importance of understanding may be growing. To be successful citizens we will need to reason about the world, we might see reasoning as the currency of the future.

For primary teachers, at present there are calls to promote reasoning (e.g. it is one of the aims of the English mathematics curriculum and is integral to Working Scientifically in the science curriculum (DfE, 2014)) but little in terms of concrete advice to make this real in classrooms and schools. Our interest in the primary science, technology, engineering and mathematics (STEM) subjects of mathematics and science has led us to consider reasoning in primary classrooms.

This paper offers a small part of the work we have been exploring on reasoning. Our goal was always to offer support for primary teachers in the classroom. Learners benefit from reflective, thoughtful teachers who constantly revisit and revise pedagogical approaches. A focus on learner reasoning, we suggest, provides a powerful direction for this self-reflective approach. Reasoning in the classroom represents a shift of perspective away from more traditional approaches of pupil learning, towards one where learners are meaning makers, reasoners and problem solvers.

## Why focus on two subjects: mathematics and science?

We have argued strongly (Cross and Borthwick, 2016) that while they are distinct subjects, mathematics and science overlap educationally in many ways. Subject boundaries, while widely recognised, are artificial. Primary teachers will be aware that many primary-aged learners do not readily acknowledge these boundaries. Both subjects require reasoning in order to solve problems, investigate, etc. We feel that primary teachers teaching both subjects are in a great position to get the very best out of the STEM subject links.

## A rationale for reasoning

We suggest that the ability to question, pose, investigate and solve problems is at the heart of mathematics and science. However, until a problem has first been understood, learners may struggle to engage with it. This presents learners with many difficulties, because there is a great deal involved in solving a problem, a fact that is often unappreciated. There are many skills involved in unravelling what a problem is about, which concepts and skills are needed and how to make use of them in finding a possible solution. Some of the required knowledge and skills will be subject specific, but there are other cognitive demands needed, such as choosing which strategy to use, how to record the answer etc. Even when learners are equipped with all of these skills and knowledge, they still need to reason which one to use or not use.

Without reasoning learners may be simply following procedures, applying rules, ignoring patterns and missing opportunities. Kilpatrick et al. (2001) offer five different strands for mathematical proficiency (which we also believe are applicable to science learning): conceptual understanding, procedural fluency, strategic competence, productive disposition and adaptive reasoning. They view these strands as equally important and interdependent. Being able to draw on each of these strands when solving problems is key to being successful, and without the reasoning strand, learners may struggle to make progress, connect ideas and knowledge and reach a level of understanding that is simply not about producing a formula or memorising a fact.

## What is reasoning?

While this paper and our interest lies with reasoning, many of those we talked to struggled to define the differences between thinking and reasoning. We felt, therefore, it was necessary to explore this further, in order to clarify what we mean when we refer to reasoning in mathematics and science.

The word thinking is used in everyday language and we use it in different ways. For example, *Anne is sure to think of a solution* (an action); *What do you think about the decision to cut down these trees to enable a housing development?* (seeking a opinion); *Why didn't you think of that before you went ahead with that idea?* (a kind of foresight).

There are many references to *thinking* in the educational literature:

> Probably the single most important lesson is that being stuck is an honourable state and an essential part of improving thinking (Mason et al., 2010).
>
> Mathematical thinking is a whole way of looking at things, of stripping them down to their numerical, structural, or logical essentials, and of analysing the underlying patterns. Moreover, it involves adopting the identity of a mathematical thinker (Devlin, 1991).

*Reasoning* is less prominent, and often, but not always, includes a reference to thinking:

> Thinking is using thought or rational judgement while reasoning is to persuade, or move by argument to express in logical form (Fowler and Fowler, 1984).
>
> Reasoning is fundamental to knowing and doing mathematics. Reasoning enables children to make use of all their other mathematical skills and so reasoning could be thought of as the 'glue' which helps mathematics make sense (Pennant et al., 2014).
>
> Adaptive reasoning refers to the capacity to think logically about the relationships among concepts and situations. Such reasoning is correct and valid, stems from careful consideration of alternatives, and includes knowledge of how to justify the conclusions (Kilpatrick et al., 2001)

We also consulted teachers about their views on what thinking and reasoning could be.

| **Thinking …** | **Reasoning …** |
|---|---|
| Thinking can be recalling facts. | Reasoning is a form of thinking. |
| Thinking can be exploring, pondering. | Reasoning is purposeful, a logical progression. |
| Thinking can be open-ended, abstract, no particular end point. | Reasoning is trying to make sense of things; it is deeper than thinking. |

| | |
|---|---|
| Thinking can be everyday thoughts, uncoordinated. | Reasoning is about logic, deeper thinking, convincing, proving, explaining. |
| Thinking can be a cognitive process that doesn't always culminate in a product. | Reasoning is the why and the how. |
| Thinking can be low level recognition or consideration of ideas. | Reasoning often uses more than one idea or concept to amend, develop or establish concept. |

In talking with teachers and reflecting about thinking and reasoning, we have settled on a definition that, for us, is a point of reference which we can draw on when exploring reasoning in primary classrooms:

> Reasoning is thinking, but it is thinking in a logical, purposeful, goal-directed way.

We have selected the words carefully within our definition. To apply some logic assumes alternative viewpoints have been considered. To have purpose implies we are focused and determined in pursuing our line of enquiry. To have a goal suggests that there is an objective in mind, but this does not have to be a final answer or a proof, this could be the process we have been engaged with or the knowledge we have gained along the way. We purposefully did not include the word 'correct' in our definition. This is because, for us, reasoning is how we navigate through a problem, to try and make sense of the situation, to select the tools to help us and to reach some sort of resolution, even if this is to continue to reason. We also believe that all learners of any age can reason, despite researchers such as Inhelder and Piaget (1958) suggesting that learners' ability to reason is quite limited until around the age of 12 years old.

So, for us, reasoning is another type of thinking, some may even say a subset of thinking. We acknowledge that the terms *thinking* and *reasoning* are often used interchangeably. On one level this is understandable and acceptable, but on another, it de-values the importance of reasoning and the skills associated with it.

## Skills of reasoning

There are many hallmarks to reasoning but one we noticed was lacking was that teachers often struggled to draw on more than a handful of the skills of reasoning. While we often talk about these skills, we did not have a point of reference to go to which collects them altogether. In working with teachers we have considered what some of the more prominent skills of reasoning might be.

| | | | | |
|---|---|---|---|---|
| convincing | working abstractly | decomposing | interpolating | evaluating |
| being logical | hypothesising | extrapolating | proving | predicting |
| deconstructing | generalising | speculating | explaining | agreeing/disagreeing |
| questioning | noticing | describing | justifying | making connections |
| recognising links | being systematic | making judgements | pattern seeking | exemplifying |
| comparing | doing/undoing | organising | refuting | tinkering |
| correcting | altering | specialising | verifying | deleting |
| offering counter examples | visualising | working backwards | wondering | puzzling |

| | | | | |
|---|---|---|---|---|
| classifying | observing | sorting | conjecturing | playing |

This collection of skills is a starting point for teachers and learners. It is not a definite list, not are the skills hierarchical, but they are interdependent, and some do demand more cognitive thinking and reasoning than others.

## How do we reason?

There are of course many ways to reason, and no one correct way. How we reason is a process we go through, often with stages intertwined and not always in the same order. One journey of reasoning may begin with some noticing, then perhaps some inner thoughts, talking and listening to others, making sense, sharing insights and finally perhaps thinking and reflecting about the reasoning that has just taken place. The way we reason is going to depend on many factors, but we want to emphasise the importance of noticing. If we do not notice something how can we begin to reason about it? To notice means we have made some observations, perhaps we have been surprised or ambushed in some way, but it has caused us to question, stop, pause, reflect, think and then reason.

## Final remarks

We suggest that a focus on reasoning is timely (with the launch of multiplication checks for eight year olds in England, for example) and hope that our work offers teachers a toolkit for reasoning.

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# Different Problem – Same Answer!

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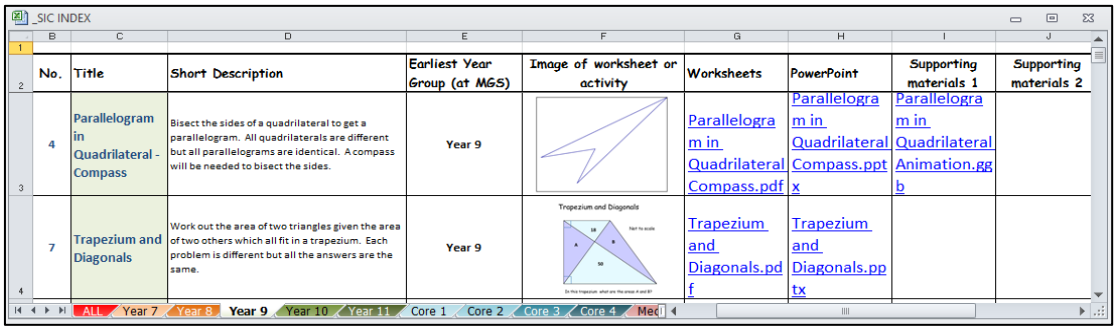
**Keywords: problem solving; invariants; proof; age 11-16; differentiation**

## Something In Common (SIC)

I have to own up to the fact that the title of my workshop was a slight misnomer. The set of resources (Burke, 2018) I have developed started out on that theme but subsequently branched out, the reason for which will become obvious. In its simplest form the resources are problem solving ones where each pupil works with essentially the same problem but with different input parameters to that of his/her neighbour. This limits any attempt to copy the answer (they can copy the method, of course) and because they all have the same answer it is easy for the teacher to check that they are correct. However, once pupils have attempted a few of these tasks they quickly catch on that the answers are all the same. So I had to broaden my remit so that the answers were all different but that the teacher would easily know what the answer was. I have achieved this by either coding the answer directly into the worksheet or by expecting the teacher to perform a simple calculation based on the input parameters. So all the answers are not all the same but they all have 'something in common' to aid the teacher.

Clearly, creating individual worksheets (12 or more is generally sufficient per class) is time-consuming but re-use and sharing makes it worthwhile. Also, it is my experience that these resources have additional benefits. There is a 'wow' factor because the pupils wonder how all the answers can be the same (or related to each other). The stronger pupils will want to work out how it was done and try to prove it with little prompting from the teacher. Thus there is also an element of differentiation.

To make the resources as easy as possible for the teacher to use the worksheets are in portable document format (pdf) files (with one, two or four worksheets to a page, as appropriate) and the originals can be found at the end of a PowerPoint presentation that holds a worked solution. There is a worked solution to every task (69 at the time of writing). Additionally, there may be supplementary files, such as Geogebra, for animation/demonstration purposes. All files are held in one folder that also contains an Excel spreadsheet used to index all of the resources. This index spreadsheet is named _SIC INDEX.xlsx and an excerpt from it is shown in figure 1.



_SIC INDEX

| No. | Title | Short Description | Earliest Year Group (at MGS) | Image of worksheet or activity | Worksheets | PowerPoint | Supporting materials 1 | Supporting materials 2 |
|---|---|---|---|---|---|---|---|---|
| 4 | Parallelogram in Quadrilateral - Compass | Bisect the sides of a quadrilateral to get a parallelogram. All quadrilaterals are different but all parallelograms are identical. A compass will be needed to bisect the sides. | Year 9 | | Parallelogram in Quadrilateral Compass.pdf | Parallelogram in Quadrilateral Compass.pptx | Parallelogram in Quadrilateral Animation.ggb | |
| 7 | Trapezium and Diagonals | Work out the area of two triangles given the area of two others which all fit in a trapezium. Each problem is different but all the answers are the same. | Year 9 | Trapezium and Diagonals | Trapezium and Diagonals.pdf | Trapezium and Diagonals.pptx | | |

ALL | Year 7 | Year 8 | Year 9 | Year 10 | Year 11 | Core 1 | Core 2 | Core 3 | Core 4 | Mec

Figure 1: An excerpt from _SIC INDEX.xlsx

The teacher has to simply click on the appropriate cell to launch the pdf (for printing), PowerPoint presentation or other resource. The tasks are also grouped by (possible) year group.

### *Trapezium and Diagonals (SIC_7)*

The first task we looked at in the workshop was Trapezium and Diagonals.

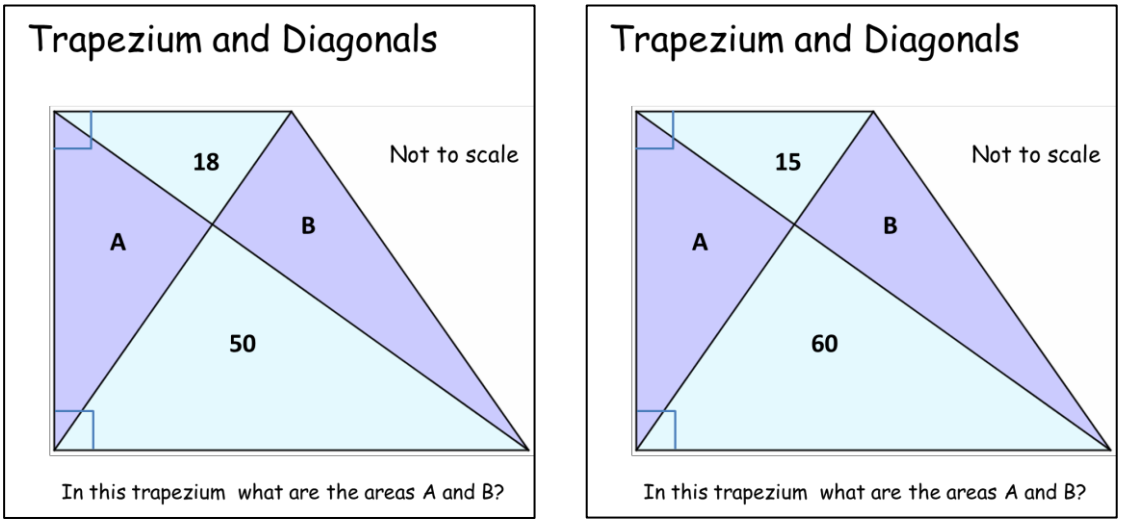



Figure 2: Trapezium and Diagonals worksheet examples.

The task is to determine the values of A and B which represent areas of triangles. It turns out that $A \equiv B \equiv 30$ for all worksheets. In fact $A = B = \sqrt{18 \times 50}$ in the first case above and $A \equiv B \equiv \sqrt{15 \times 60}$ for the second. Each worksheet is designed so that the product of the two numbers is 900 and hence all answers are 30. The inspiration for this task came from Ball (2005) and readers may spot the striking resemblance to the book's front cover. I solved the delightful problem in the book and decided to look at the general case which gave me an invariant (the geometric mean of two numbers) that I could exploit to make individual worksheets.

### *Sine Rule Discovery (SIC_2)*

In this example, each pupil is given his or her own triangle from which to calculate the sine rule ratio. An example of two worksheets is shown in figure 3.

All triangles should produce a sine rule ratio of close to 14.5 cm. All the triangles are different so how can the ratio be the same (for the whole class)? If each triangle was enlarged by a scale factor of 2 then we would expect the ratio to double also, so it must mean something. Then I ask the pupils to draw the circumcircle (with appropriate scaffolding from me) with pairs of compasses and then to measure the diameter. The diameter is close to 14.5 cm.

The PowerPoint presentation shows proofs for the sine rule and the extension to reveal $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$= 2R$, where $R$ is the radius of the circumscribed circle.

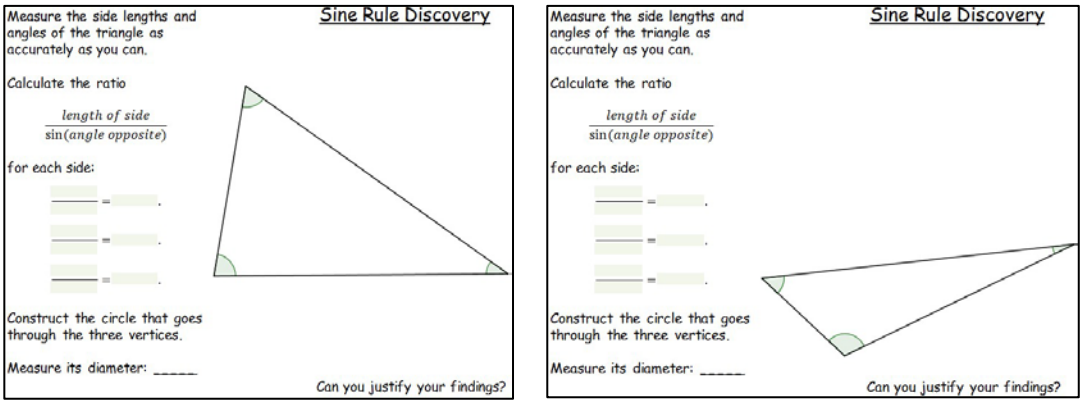



Figure 3: Sine Rule Discovery worksheet examples

The final result then explains how all of the sine rule ratios in the classroom are the same – all the triangles were drawn using the same circle.

### *Polygon in Annulus (SIC_18)*

This worksheet (see figure 4) differs from the others in that it asks each pupil to solve two problems. The hope is that they then try to understand more deeply what is going on (since both answers are the same). The task is deceptive because you do not need to work out any angles and so trigonometry is not required. All that is required is Pythagoras' theorem.

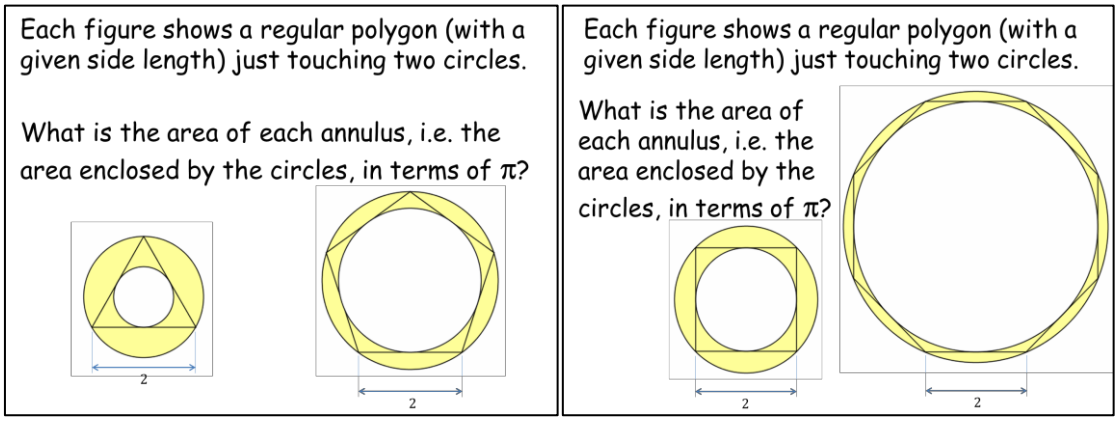



Figure 4: Polygon in Annulus worksheet examples

The motivation for this task came from the classic problem that asks what single measurement is required to determine the area of an annulus. The only measurement you require is the length of the chord that is tangent to the inner circle. If we call this length $l$ then the area is $\pi\left(\frac{l}{2}\right)^2$. In all our cases $l = 2$ so the area of every annulus is $\pi$.

Pupils find this hard to believe and so I have made a number of the rings from acrylic (with the polygons etched on to them) which could be weighed to prove the point. There is also a Geogebra animation, which is useful to show that the area is equal to a circle with the chord as its diameter (see figure 5).

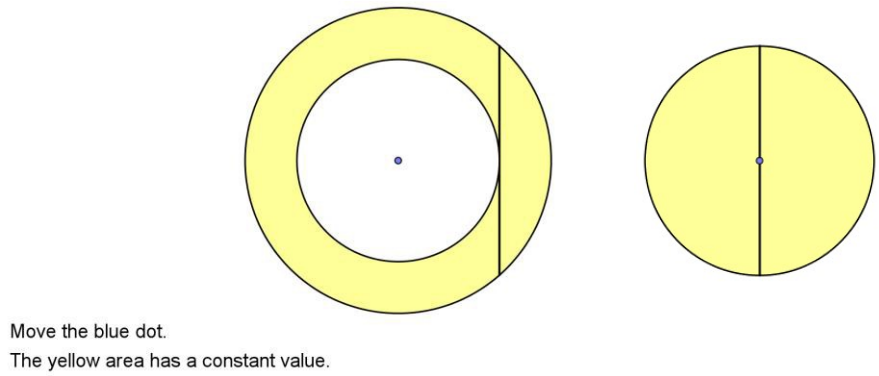



Figure 5: Polygon in Annulus Geogebra demonstration examples.

The demonstration shows that the polygons were a complete red herring (as were the right angles in the Trapezium and Diagonals task). They are, of course, deliberate on my part and meant to stimulate discussion at the end as to what was relevant to the solution.

### *Further Tasks*

Other resources cover topics such as surds, sequences, differentiation, integration, vectors, functions, coordinate geometry, probability and mechanics. Some combine two topics. For example *Leonardo of Pisa* (SIC_21) gives the $n^{th}$ term of a sequence that incorporates surds and the pupils have to determine specific terms in that sequence. All of the terms are integers, which is the first surprise. The second is that it turns out to be the Fibonacci sequence (hence the curious name of the task). The pupils are then asked to prove that the ratio of consecutive terms tends to the golden ratio.

### Summary

This article gives a brief insight into the collection of resources I have called *Something In Common* (Burke, 2018). Some of the resources are the result of re-packaging resources I have found on various websites (especially NRICH), books, textbooks, ideas from colleagues and some are of my own creation. I hope that I have given credit where it is due in my resources. I also hope that teachers and educators will use the resources as they see fit in their teaching. I have provided all the source materials needed for anyone to adapt them for their own purposes – because they prefer their own worked solution, or to create worksheets for visually impaired pupils, or to translate into their own language, etc.

I welcome all feedback on how to improve the resources and would welcome ideas for new tasks.

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# How did Euclid and Archimedes manage without calculus?

Bob Burn

**Keywords: history of mathematics, Euclid, Archimeded, calculus**

## Introduction

Euclid and Archimedes established a chain of results about areas and volumes. However the Paradoxes of Zeno (which we find in Aristotle) made any use of infinite processes suspect, and these have been avoided by both Euclid and Archimedes. This rules out limits and calculus as we know it today.

Archimedes credited Eudoxos (c.350 BC) with the method used by Euclid and himself. It depends firstly on the notion of ratio and in particular what we call today the Axiom of Archimedean order, that if two quantities $a$ and $b$ are given, with $b < a$, some multiple of $b$, say $nb > a$. In modern terms, this condition makes the number field contain the rational numbers, but not more than the reals (so no infinitesimals).

Euclid X.1 was an often used equivalent to Archimedean order, that if two quantities $a$ and $b$ are given with $b < a$, then when $a$ is halved a sufficient number of times, the result is less than $b$.

The essential method for avoiding limits was that of double contradiction. To prove $a = b$, we may prove that both $a > b$ and $a < b$ are false.

Please see the file available on the BCME website for the theorems of Euclid XII and for Archimedes' treatment of spirals, which was richly generalised during the 17th century.

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# The pedagogy of an asynchronous online course: supporting students' engagement

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**Keywords: asynchronous; online; active learning**

As part of the MA Mathematics Education programme at UCL IoE, we offer an online module that focuses on the potential digital technology could have in supporting the teaching and learning of school mathematics. We outline and reflect on the emergence of the online pedagogy of the tutors, as we designed, then teaching of this module.

## The context

The *Digital Technologies for Mathematical Learning* module is the latest optional module added to the MA in Mathematics Education programme. First taught in 2014, this module has two e-learning aspects: its online delivery, and the e-focus of the course itself, consisting of i) familiarisation with a wide range of digital tools and resources (graph plotters, dynamic geometry environments, statistical software, fully interactive online packages) and ii) critical reflection on the implications of using such tools in the learning and teaching of mathematics at secondary school level (11-18 years old students). The main aim of this course is to encourage participants to reflect critically on the potential and limitations of digital technologies for the learning and teaching of mathematics by providing opportunities for participants to apply knowledge of relevant research and theory in their professional contexts.

The module is asynchronously delivered online, with learning resources that facilitate information sharing beyond the constraints of time and place amongst the participants. Designing an online module was a new experience for us, the tutors (the author and one other colleague, both co-designers and co-leaders of this module). In doing so, we were encouraged by the research into effective online instruction (Maki and Maki, 2007) which offered three conclusions:

1) online instruction can be as effective as traditional instruction;
2) to do so, online courses need cooperative/collaborative active learning *and*
3) strong instructor presence.

We started from the premise that we would be able to fulfil the 'strong instructor presence' requirement (even if we were not quite sure what it meant when we started). Hence, in designing the module our main concern was how to facilitate and support students' cooperative and collaborative active learning, given the constraints and affordances of the asynchronous online module. We quickly realised that we needed to re-conceptualise what active learning meant in an asynchronous online environment and then find ways to promote and support it.

## Our design principles: a framework

Our review of the literature provided us with a useful framework for implementation of active learning practices into an asynchronous online environment, referred to in literature as 'A three-part approach: An Architecture of Engagement (Riggs and Linder, 2016) consisting of: Element 1: Syllabus Communication and Engagement Policy, Element 2: Course Orientation, and Element 3: Modular Course Structure.

In the following, we illustrate how we used this framework in the design of our course, describing the elements above in an order that makes is easier for the reader to follow, rather than describing our moving forwards and backwards between these elements.

### *Architectural Element 3: Modular Course Structure*

> A modular course structure helps to frame the architecture of engagement throughout the course. By modular course structure, we mean dividing the course chronologically with multiple units, with each module containing all of the course materials, learning activities, assignments, and assessments for that unit. (Riggs and Linder, 2016: 4)

In following this framework, we divided the curriculum for this module into three themed sections: Visualising, Generalising and Expressing, and Modelling, with each theme lasting for three weeks. During each of the themed sections, the course curriculum is arranged into a series of short tasks that culminate in the main task of designing and trialling a learning activity relevant to each theme. These short weekly tasks are signposted on the virtual learning environment of the course (Moodle) at the beginning of each week and include both offline and online tasks.

Each theme ends in an activity week for which participants are required to: choose a software tool relevant to the theme, design a learning activity using features of good practice identified from the literature, use it in the classroom or another learning environment and evaluate its use with reference to teaching and learning of mathematics and connections with the literature.

In each theme, at least one task will form the basis of an online group discussion, where such contributions include for example, written observations on views and perspectives of fellow participants. The tutors also contribute to these discussions, with the aim of encouraging informed reflection and raising critical awareness of research literature.

In our yearly end of themes and end of year surveys, the students are invited to share their experience of studying this online module. Their comments suggest that our modular course structure does helped them in a number of ways by:

- pacing the learning experience to prioritise activities and to help prevent participants from feeling overwhelmed:

  > At first, I was afraid I wouldn't get the proper guidance, but now I believe that the structure of the course is pretty clear and I know what I have to do every week. (ET, 2018)

- discouraging procrastination by providing regular milestones and deliverables:

  > I find the module really interesting. I really like the fact that we have weekly tasks to complete. I find the timeframe given to complete the weekly tasks to be appropriate especially for a working student. (TC, 2018)

- providing opportunities for guidance and feedback on set tasks:

  > I think that these tasks [and the feedback received] help me improve my writing skills and how to critically reflect on the bibliography. (BN, 2018)

### *Architectural Element 2: Course Orientation*

> An orientation for an online asynchronous course should introduce students to the structure of the course. (Riggs and Linder, 2016: 3)

Each week the tutors upload: LOs (learning objectives) and description of the week's content: readings (essential and indicative) and the weekly tasks, which are tightly structured towards the course aims and learning objectives. This structure is followed every week and proves very helpful to the students:

> At first, I was afraid I wouldn't get the proper guidance, but now I believe that the structure of the course is pretty clear and I know what I have to do every week (ET, 2018).

### *Architectural Element 1: Syllabus Communication and Engagement Policy*

> The asynchronous online course syllabus must do all the foundational things a face-to-face course syllabus does, but it must also: set communication policies and expectations for online engagement; and set a course schedule that outlines the frequent and meaningful engagement. (Riggs and Linder, 2016: 2)

Indeed, our expectations of studying an online module were made explicit to the participants, as exemplified below:

- How much time should be set aside weekly to spend on the module?

> When studying online, you should expect to log in for short periods several times each week. It is very important to keep to the timetable of each week or you risk falling behind and missing crucial discussions (which can easily happen!). (module handbook)

> Given that this is a 30-credit module, it is expected that you will spend 15 hours per week studying for this module, and the level of work is designed to reflect this. (module handbook).

- What does attendance mean?

> For the online modules, online presence and task completion are accounting for your attendance. You are expected to complete activities for and contribute to at least 70% of the required online activities. […] Your attendance (online contributions and task completion) will be monitored by the module tutors on a weekly basis. (module handbook)

## Supporting the architecture of engagement

Research (Garrison, Anderson and Archer, 2001; Lewis and Abdul-Hamid, 2006 and Russo and Campbell, 2004 cited in Dixson, 2010) indicates that it can be helpful for instructors of online asynchronous courses to create an architecture for their own engagement, too. As tutors, we made sure that we had social presence demonstrated on a weekly basis through: uploading materials, posting messages, guiding student learning, providing feedback at set times, sending reminders, contributing to some online discussions.

Whilst we strove to maintain a strong online presence, we tried to engage minimally in the discussions, as we did not want to deter the students from participating. However, the participants were keen to receive individual feedback from the tutor:

> It would be good if the tutors could provide more detailed and personalised feedback to the tasks we have handed in each week. At least each individual of us will be clear if we are on the right track of the task objective. (SA, 2017)

In order to address our participants' need to communicate with tutors, we:

- offered personalised feedback on tasks at the end of a theme only;
- offered general comments on all the other tasks, raising common issues, etc;

as well as

- encouraging and prompting discussions; and
- providing a time frame for replies to questions posted on discussion forums.

### *Re-conceptualising active learning in an asynchronous online environment*

Since there was no direct teaching for our online course, we designed active learning to take place as the result of doing short, manageable tasks. For example, we included offline tasks such as designing and trialling mathematics lessons involving specific software that bridge learners' interactions with digital media and the mathematical concepts, followed by online contributions where they would reflect on their experience.

### *Supporting cooperation and collaboration*

As designers, we created opportunities for participants to interact. After the first presentation of the course, we realised we needed to secure engagement too. The course re-design included:

- one task that formed the basis of an online group discussion;
- tutor-allocated groups for collaborative tasks;
- Pick-a-Paper tasks, where participants choose one paper and its commentary from a participant in their group and post their comparative remarks and reflections.

Whilst these strategies ensured engagement with tasks, the contributions were not always very insightful. Some participants say:

> I feel like I cannot effectively and efficiently discuss with other participants of this course. Although there is a forum to discuss, I feel like it is better for me to discuss face to face with them. I am not comfortable with discussing online. (IN, 2018)
>
> But, the truth is that the forum is impersonal and writing actually makes you feel anxious because it is formal. (BN, 2018).

## Some conclusions and next steps

We have become interested in why some participants are more engaged than others and for the future we would like to investigate further how to support all of our participants to become active learners in their asynchronous online experiences.

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# How might the Numberlink Board™ be used to develop deep conceptual understanding of multiplication through exposing structure and making connections?

Katie Crozier

**Keywords: Numberlink Board; learning multiplication; multiplicand; structure; double number line**

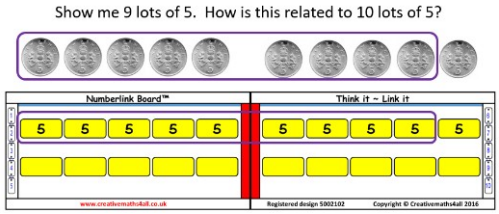
## Introduction

The extent to which deep conceptual understanding and visualisation of multiplication can be developed through exposing structure and making connections, was an action research project that I conducted for my Masters degree. My research questions focussed on finding out about how children within my year 4 class developed multiplicative reasoning. Mathematics teaching pedagogies that aim to promote deep conceptual understanding and visualisation of multiplication as repeated aggregation were considered in relation to the data collected. This paper provides a brief overview of the use of the Numberlink Board representation and the extent to which it supports children's understanding of the structure of multiplication and their ability to apply multiplicative reasoning.

Having observed many children struggle with inefficient methods of recalling multiplication facts, for example finding nine lots of something by counting up from one lot, I wanted to strengthen the awareness of the distributive law and encourage children to use key facts to derive other facts. With regard to the recall of multiplication facts, I do think that accurate and rapid recall of facts is important but that it should not be a stressful and negative experience. Recall of number facts based on structure and number sense is far more powerful (Boaler, 2016). Brain researchers who studied the learning of mathematics facts in two ways, through strategies and through memorisation concluded that both pathways are effective at recalling facts fluently but that those who learned through strategies, for example learning $17 \times 8$ from $17 \times 10$ and then subtracting $17 \times 2$, were able to connect their conceptual understanding to new problems (Delezar et al, 2005). In their research to analyse children's different approaches to arithmetic Gray and Tall (2007) found that the more successful children used a lot more derived facts and the least successful children resorted to counting to reach each answer.

## Using the Numberlink Board to expose the multiplicand

I designed the Numberlink Board so that the multiplicand was a key feature of the board, to emphasise the number in the repeated group or set. In other representations, like the counting stick, this part of the multiplicative structure was implicit, it was stored mentally rather than shown on the representation. I argue that this is the essential feature of the repeated addition structure of multiplication and as such needs to be shown explicitly. I worked with the children in the research project to strengthen visualisation of repeated addition with relation to multiplication. I designed the board so that the middle is represented by a large red line - a key reference point on the board to highlight where 'five times' the multiplicand is. Before thinking about products, I asked children to explore the representation of ten lots of a particular multiplicand both cardinally and symbolically, as shown in Figure 1 below with associated questions.




Show me 10 lots of …

Show me 9 lots of …

How did you find 9 lots so quickly?

How is 5 lots related to 10 lots?

Show me 5 lots, now show me 6 lots

Figure 1: The Numberlink Board with multiplicands represented

I found that spending time on the orientation of finding multiplicands in relation to the key points of '10 lots' and '5 lots' laid the foundations for using the distributive law. Children explained that 6 lots was one more than 5 lots and 9 lots was one less than 10 lots. When putting the products onto the Numberlink Board I asked the children to place them in the order one lot, ten lots, then five lots (which we discussed could be found by halving ten lots). Once these key facts had been written on the board, other products could then be derived by looking at their relative position. Figure 2 below shows how the three parts of repeated addition multiplication are exposed on the Numberlink Board.

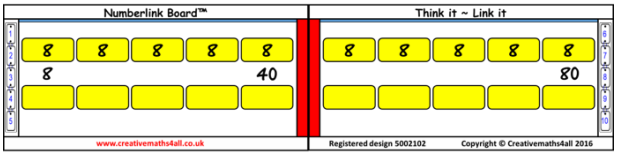



Figure 2: Numberlink Board showing the structure of multiplication and key facts

The children started to derive and prove multiplication facts, using the Numberlink Board as a visual support, for the application of the distributive law as demonstrated by the children's comments below:

5 × 6: the answer is 30 because $\frac{1}{2}$ of 60 is 30

9 × 6: you take away 6 from 60

6 × 6: it is 36 because 5 × 6 = 30 and it's one more 6

Another stimulus for exposing the multiplicand came when a child was trying to work out 98 × 3. I asked her to give me an approximate answer; she considered this for a short time and said 'about 500'. We discussed what 98 × 3 meant and when encouraged to write down 98, 98 and 98 on her piece of paper she then reasoned that the answer would have to be about three hundred and even clarified that it had to be a little less than three hundred, as 98 was a little bit less than 100. By reminding her that multiplication was about repeated addition, her number sense was engaged.

## The significance of the double line representation in developing multiplicative reasoning

The relationship of numbers along the line of ten blocks on the Numberlink Board has been discussed in relation to the distributive law. The second row of blocks on the Numberlink Board was added initially so that children could explore place value links and scaling by 10 or 100, for example children explored how multiples of 8 were linked to multiples of 0.8, or 80 or 800. As the action research spiralled and the use of the Numberlink Board developed, the children were encouraged to use the second row of blocks to spot more patterns and connections between the rows as well as along the row of products.

### *Using the second row of the Numberlink Board to adjust from known facts, estimate and develop number sense*

I encouraged some children in the class to apply their existing understanding of multiplication facts to go deeper and think about connected facts. In the weekly multiplication fluency sessions some children started with their base facts, for example working on the multiples of eight, they would use the second row to scale up or down and find the multiples of 80 or 0.8, as shown in Figure 3. We spent time discussing how and why each product was ten times bigger or smaller than the multiples of eight.



Figure 3: Making connections on the second row by scaling up or down

I developed this idea further using the second row of blocks on the Numberlink Board to estimate products to multiplication calculations. I found that by exposing both the actual multiplicand and the rounded multiplicand, children were able to think about a reasonable answer as shown in Figure 4.



Figure 4: Using the second row of blocks to support estimation and adjusting from known facts

Some children were then able to go further and see how many times the multiplicand had been rounded up or down and by how much, so that they were then able to mentally work out the answer. This was shown by jottings on whiteboards; 50 × 6 = 300, 52 × 6 = 300 + 12 = 312

I also used the second row on the Numberlink Boards to address a common misconception with regard to money. When answering £1.99 × 6, many children calculated £11.99. For many of these children, the error had occurred not through the rounding; they had rounded £1.99 to £2 and then multiplied this 6 times. The point of difficulty was in understanding that adjusting back needed to be 6 × 1p not just 1p – the picture of adjusting back six times was not clear. Using the second row of the boards, as shown below in Figure 5, children were able to see each multiplicand, which supported understanding of why 6p needed to be taken away from £12, as they could see how many times the multiplicand had been rounded up.

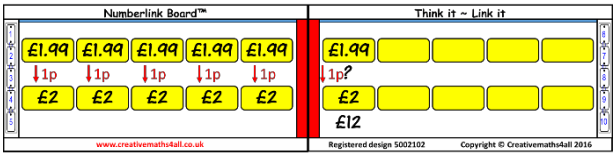



Figure 5: Exposing common misconceptions: Is £1.99 × 6 equal to £11.99?

During the research project, I also explored using the second row of blocks to expose the structure of multiplication using procedural variation. The effect on the learner of using procedural variation is analysis of the structure of the calculation and a deeper understanding of the concept (Lai and Murray, 2012). I explored with children how the structure of the multiplication calculation changes if the multiplicand is increased or decreased by one and how this affects the product, for example:

46 × 6 = 276, how can we use this to work out 47 × 6?

This requires the children to use one known product and adjust the answer to reflect the change in multiplicative structure. These examples are shown on the Numberlink Board in Figure 6.

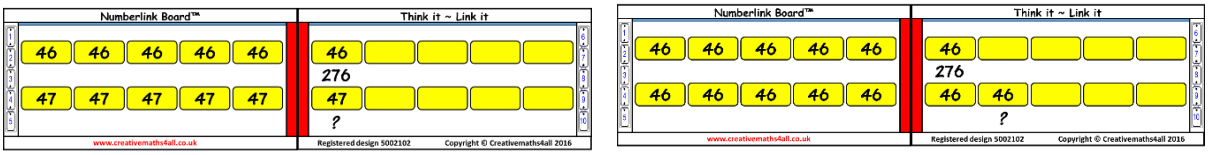



Figure 6: Revealing the structure when the multiplicand or multiplier is changed by 1.

## Conclusion

At the end of the research period, results indicated that children had an increased awareness of the structure of multiplication and could apply the distributive law to derive multiplication table facts. Children used the double line on the Numberlink Board to reason mathematically about patterns and connections between multiples, there was limited use of the distributive law when multiplying with larger numbers.

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# How do online professional learning courses compare with face-to-face?

Sue de Pomerai and Sharon Tripconey

*Mathematics in Education and Industry (MEI)*

**Keywords: professional development; online learning; teacher education; teacher educators**

## Background and context

The Further Mathematics Support Programme (FMSP), managed by MEI has been working to ensure universal access to A level Mathematics and Further Mathematics in England since 2005. However, there are still barriers to schools providing access to these qualifications. For example, a shortage of teachers with appropriate skills at the required level. Since the inception of the FMSP, the number of students studying A level Mathematics has increased by 88% and A level Further Mathematics by 204%, which has further increased the demand for teachers to acquire the necessary skills and experience to teach mathematics at this level.

Over the last 10 years, the FMSP has developed a variety of professional development (PD) courses, mainly aimed at teachers of A level Mathematics and Further Mathematics, with the aim of enabling better access to Level 3 mathematics PD. The range of courses offered has included traditional one day face-to-face courses, series of one- to two-hour twilight sessions, series of weekly one- to two-hour live online courses and sustained blended courses that take place over six months and consist of both face-to-face days, some on Saturdays and some on weekdays, and online sessions. The number of participants on each type of course in the academic year 2016/17 are shown in table 1.

| Type of course | Number of courses offered | Number of participants 2016/2017 |
|---|---|---|
| Live, online (after school hours) | 19 | 96 |
| Sustained (blended six month courses) | 8 | 195 |
| One day face-to-face (local, week | 83 | 1123 |
| Twilight (local, week days) | 45 | 549 |
| Introduction to new A level days | 52 | 1139 |
| Totals | 207 | 6402 |

Table 1 Participation in different types of level 3 mathematics PD provided by FMSP 2016/17

In spite of the large number of teachers undertaking professional learning courses with the FMSP, there has been some indication in recent years that it is becoming increasingly more difficult for teachers to access PD during the school day.

## Results of initial survey

In our continued efforts to improve teachers' access to PD, we asked FMSP course participants to express a preference for one or more of the types of courses that we currently provide. Teachers were asked to complete the statement, '*It would be easier for me to attend CPD events if they were…*' and prompted to select all options that applied. Collated responses are shown in table 2.

| Type of course | Number of participants in sample | on a school day | on a Saturday | as one or more twilight events | offered through a series of sessions in an online classroom |
|---|---|---|---|---|---|
| **Live, online (after school hours)** | (20) | 4 (20%) | 2 (10%) | 6 (30%) | 16 (80%) |
| **Blended (Saturday study day)** | (54) | 8 (15%) | 48 (89%) | 9 (30%) | 27 (50%) |
| **Blended (weekday study day)** | (26) | 26 (100%) | 4 (15%) | 1 (4%) | 5 (19%) |
| **Local one day (Face to Face weekday)** | (53) | 50 (94%) | 4 (8%) | 9 (17%) | 6 (11%) |
| **Twilight (local weekday)** | (32) | 16 (50%) | 5 (16%) | 21 (66%) | 4 (13%) |

Table 2 Responses to survey*: 'It would be easier for me to attend CPD events if they were…'*

It can be seen from the highlighted cells that, unsurprisingly, a respondent typically included the same type of course that they had attended as one of their preferences. For example, teachers who attended an event on a school day indicated that it would be easier to attend PD on a school day. We then went on to look in more detail at the responses for different courses

### *Live online*

Online courses, which are scheduled after the school day, was selected as a preferred choice by 16 (80%) of the teachers who took an online course. Of these, just under half gave it as their only choice. It would be interesting to find out whether this is a learning preference or whether other factors influence their choice. Of the remaining four, who did not select online as a preferred option, three said they would prefer twilight sessions and one teacher chose Saturday, which may indicate that these teachers are not able to or do not wish to take time out during the school day.

### *Sustained blended courses*

These courses are sustained over a period of six months and consist of a blend of online sessions and face to face study days. The study days are either scheduled on a week day or a Saturday. Considering the blended courses that had Saturday study days we found that 48 of the 54 who chose the courses with Saturday study days gave Saturday as a preference, of which 20 ticked no other option, which is interesting given that online sessions are a key feature of a blended course. Although attending a Saturday study day, only three of the 54 expressed a preference for study days on a school day, which may suggest that they were unable to attend the face-to-face element of the course on a week day.

Comparing this to the blended courses that had weekday study days we found that all participants gave school day as a preference and for 18 of the 26 that we surveyed, this was their only preference. For this group five selected online as one of their options but no one gave it as their only option.

### *Face to Face local courses*

Traditional one day PD courses held locally on school days are still a popular choice for many teachers but there are an increasing number of teachers that attend local

twilight PD sessions. Considering the sample of 53 teachers that attended a one day course on a week day, only three selected Saturday as one of their choices. One teacher said they would consider a Saturday if they were given time off in lieu.

Two thirds of the teachers who attended local twilight sessions included twilight sessions in their preference, half of these gave it as their only option. The results from the 85 teachers attending local face to face courses show preference for PD on a school day, which seems to indicate that this group were not inclined to, or not able to undertake professional learning at weekends. It is interesting to note that only 10 of these 85 teachers included online courses in their preferences.

***Summary***

It would appear that teachers who have already engaged with online learning, either through live online or blended courses, tend to include it in their PD preferences. However, the blended course sample, that chose to attend weekday study days rather than Saturday, showed only a slightly greater preference when compared to teachers taking one day courses or face to face twilight sessions. This leads us to speculate as to whether choosing the weekday study day indicates a preference for a more traditional course approach from this group of teachers despite being enrolled on a blended course.

The questions we would like to explore further are why teachers attending face to face courses have not engaged with online learning. It could be that they feel their needs are fulfilled by the local provision, they do not fully understand what an online course involves or it could be that they are uncertain as to whether an online course can fulfil their needs

## Looking more closely at TFM

Teaching Further Mathematics (TFM) is one of a suite of sustained, blended courses offered by the FMSP. The course has two units, TFM1 and TFM2, each of which consists of two face to face study days, a series of live online sessions and access to web-based resources and forums. The online sessions are recorded so that if a teacher is unable to attend a live session, or if they wish to revisit material, they can playback the recording.

73 teachers, attending a study day (Friday or Saturday), were asked to complete a brief survey to consider the advantages and limitations of online learning and face-to-face learning. The survey consisted of three questions

1. Which of the following best describes your preference?
    - *I prefer to attend the online sessions (live)*
    - *I prefer to watch recorded sessions*
2. What are the advantages / disadvantages (if any) of online course sessions?
3. What are the advantages/disadvantages (if any) of face to face course days?

There were 43 responses (59%) that indicated a preference to attend live online sessions, 24 (33%) preferred to listen to the recordings and 6 (8%) had no preference.

Considering responses from the group who preferred to attend online sessions conducted live; 13 of the 43 said that being able to review content and catch up from recordings was an advantage, 18 liked being able to ask questions and get immediate feedback and 12 said that the sessions gave structure to the course and helped keep them on schedule. It was interesting that three people said they felt more confident giving responses online. Several cited the lack of travel and cover expenses as an advantage, which reflects the results of earlier papers (de Pomerai et al. 2009, 2011).

When asked about disadvantages, 19 out of 43 said that the scheduled day for online sessions might not be convenient for everyone. This is an inevitable problem given that sessions are timetabled early evening and many people, particularly those with young families, often have other commitments. Other notable comments related to lack of discussion between participants and being unable to influence the pace of the sessions.

The responses from the group who prefer to view the recordings were similar to those that preferred live sessions: almost half commented on the advantage of no travel or expense however, the largest number of responses, 15 of the 24, related to the flexibility such as being able to view the recordings when convenient, being able to speed up and slow down or pause to give time to answer questions.

> I find live sessions a bit fast for me so I watch recordings and pause them
>
> I'm not a social person and face to face makes me uncomfortable (especially on topics I struggle with, I don't want to appear stupid)

Lack of interaction was given as a disadvantage, along with finding it difficult to ask for help. When asked about the advantages of face to face sessions the overwhelming advantage given by both groups was meeting and working collaboratively with others. They stated that they enjoyed active learning, problem solving and being able to share ideas with others.

> Great fun working on problem solving with others
>
> Forces you to take time out of the classroom and focus on your own learning

## Conclusion

Whilst some teachers find it difficult to get released from school, others are still able to get permission to attend professional learning courses, particularly at a time of curriculum change. This is evidenced by the fact that over 1100 teachers in England were able to attend one of the 52 'Introduction to new A level' days that were organised by the FMSP in Autumn 2016 and Spring 2017 (see table 1).

There were no major surprises in the responses, teachers who attend on a Saturday tended to say they prefer Saturday; the fact that they are there already gives an indication of their preference. What is more interesting is the teachers who engage in PD despite the courses being inconvenient for them. What this paper cannot comment on is the number of teachers who do not attend any FMSP PD, how could we find out why they do not attend?

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# Dynamo Assessment - what it may tell us about primary school children's mathematics

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**Keywords: mathematical development; mathematical difficulties; assessment; primary school children; number concepts; arithmetic**

## Background

Dynamo Assessment is a computer-based assessment that takes account of components of early numeracy. This assessment has so far been predominantly used for diagnosing the components suitable for intervention in children with mathematical difficulties (see https://dynamomaths.co.uk/ for further details). Dynamo Assessment identifies the mathematical components with which children are struggling providing a detailed profile to inform intervention. Interventions based on the assessment in Malta have shown promising initial results (Zerafa and Dowker, 2016).

The subtests may be grouped into Number Meaning, Number Magnitude and Number Relationships, adding the scores for the tests in each group.

- Number Meaning includes visual quantity approximation, single-digit number, counting and multi-digit number.
- Number Magnitude includes number comparison, estimation, ordering numbers and number sequencing.
- Number Relationships includes number facts, mental strategies, number bonds, problem solving, time and multiplication.

## Method

Participants were 2385 children from London primary schools, with which the second author had contact. They were recommended for testing as their teachers considered them to have mathematical difficulties. They included 280 seven-year-olds; 627 eight-year-olds; 594 nine-year-olds; 526 ten-year-olds; and 358 eleven-year-olds. The children were from 318 different schools. There were 52% male and 48% female.

All of the 2385 children were independently supervised by school teachers or support teachers as they completed the Dynamo Assessment.

In this study, there were two aspects of the three groups of tests that were analysed:

(1) Each child's combined scores on the tests in each group.

(2) The combined mean reaction times of each child to the tests in each group. (Only reaction times for correct answers were included.)

## Results

Pearson correlations were computed between Number Meaning, Number Magnitude and Number Relationships combined scores. All correlations were significant at the 1.1 level.

| | Mean **Number Meaning** Combined Score out of 20 | Mean **Number Magnitude** Combined Score out of 20 | Mean **Number Relationships** Combined Score out of 30 |
|---|---|---|---|
| 7-year-olds | 14.04 (s.d. 5.2) | 12.18 (s.d. 4.74) | 9.54 (s.d. 6.98) |
| 8-year-olds | 14.54 (s.d. 1.66) | 11.78 (s.d. 4.75) | 9.78 (s.d. 7.15) |
| 9-year-olds | 18.65 (s.d. 2.11) | 12.87 (s.d. 5.18) | 10.01 (s.d. 6.74) |
| 10-year-olds | 18.99 (s.d. 1.76) | 14.04 (s.d. 6.63) | 19.41 (s.d. 1.35) |
| 11-year-olds | 19.3 (s.d. 1.39) | 14.54 (s.d. 5.48) | 19.28 (s.d. 7.1) |

Table 1: Mean Number Magnitude, Number Meaning and Number Relationships Combined Scores by age group

A Between Participants Analysis of Variance was carried out, with Age Group as the between-participants factor, and Number Meaning Combined Score, Number Magnitude Combined Score and Number Relationships Combined Score as the dependent variables.

There was also a highly significant effect of Age on Number Meaning Combined Score ($F(4, 2381)= 13.26$; $p = 0.001$). Tamhane 2 post-hoc tests showed that there were no significant differences between 7- and 8-year-olds, 8- and 11-year-olds, 9- and 10-year-olds, 9- and 11-year-olds or 10-and 11-year-olds. There were, however, highly significant differences between 7- and 9-year-olds, 7- and 10-year-olds, 7 and 11-year-olds, 8- and 10-year-olds, 9- and 10-year-olds and 9- and 11-year-olds. All significant differences were in the direction of older children scoring higher.

There was a highly significant effect of Age Group on Number Magnitude Combined Score ($F(4, 2381)= 24.467$; $p = 0.001$). Tamhane 2 post-hoc tests showed that there were no significant differences between 7- and 8-year-olds, 7- and 9- year-olds, or 10- and 11-year-olds; but there were highly significant differences between 7- and 10-year-olds, 7- and 11-year-olds, 8- and 9-year-olds, 8- and 10-year-olds, 8- and 11-year-olds, 9- and 10-year-olds and 9- and 11-year-olds. All significant differences were in the direction of older children scoring higher.

There was also a highly significant effect of Age on Number Relationships Combined Score ($F(4, 2381)= 12.86$; $p = 0.001$). Tamhane 2 post-hoc tests showed that there were no significant differences between 7- and 8-year-olds, 7- and 9- year-olds, 8- and 9-year-olds or 10- and 11-year-olds. There were, however, highly significant differences between 7- and 10-year-olds, 7- and 11-year-olds, 8- and 10-year-olds, 8- and 11-year-olds, 9- and 10-year-olds and 9- and 11-year-olds. All significant differences were in the direction of older children scoring higher.

Pearson correlations coefficients were computed between Number Meaning. Number Magnitude and Number Relationships *combined reaction times*. The reaction times were also correlated with the scores for these groups of tests correlated significantly with one another and with the test scores ($p < 0.01$ in all cases).

Number Meaning Combined Reaction Time correlated negatively with Number Relationships Combined Reaction Time, but the other reaction time correlations were positive. For Number Meaning and Number Magnitude, the combined reaction times correlated negatively with the combined scores (high performance was associated with speed). Unexpectedly, however, Number Relationships Combined Score correlated positively with Number Relationships Reaction Time.

| | Mean **Number Meaning** Combined Reaction Time | Mean **Number Magnitude** Combined Reaction Time | Mean **Number Relationships** Combined Reaction Time |
|---|---|---|---|
| 7-year-olds | 103.93 (s.d. 78.46) | 291.95 (s.d. 171.19) | 690.78 (s.d. 422.62) |
| 8-year-olds | 87.18 (s.d. 60.56) | 301.53 (s.d. 154.06) | 786.57 (s.d. 455.76) |
| 9-year-olds | 70.23 (s.d. 47.97) | 261.21 (s.d. 147.3) | 874.75 (s.d. 433.82) |
| 10-year-olds | 61.48 (s.d. 38.34) | 250.93 (s.d. 177.97) | 931.18 (s.d. 472.88) |
| 11-year-olds | 59.11 (s.d. 38.56) | 215.8 (s.d. 166.99) | 889.31 (s.d. 435.98) |

Table 2 Mean Number Magnitude, Number Meaning and Number Relationships Combined Reaction Times by age group

A Between Participants Analysis of Variance was carried out, with Age Group as the between-participants factor, and Number Meaning Combined Reaction Time, Number Magnitude Combined Reaction Time and Number Relationships Combined Reaction Time as the dependent variables

There was a highly significant effect of Age on Number Meaning Combined Reaction Time ($F(4,2381)= 37.574$; $p = 0.001$). Tamhane 2 post-hoc tests showed that all differences between ages were significant, except between 10- and 11-year-olds. All significant differences were in the direction of older children having shorter reaction times.

There was a highly significant effect of Age on Number Magnitude Number Magnitude Combined Reaction Time ($F(4, 2381)= 14.99$ $p = 0.001$). Tamhane 2 post-hoc tests showed that there were no significant differences between 7- and 8-year-olds, 9- and 10-year-olds, 9- and 11-year-olds, or 10- and 11-year-olds. However, there were significant differences between 7- and 9- year-olds, 7- and 10-year-olds, 7- and 11-year-olds, 8- and 9-year-olds, 8- and 10-year-olds, 8- and 11-year-olds and 9- and 10-year-olds. All significant differences were in the direction of older children having shorter reaction times.

There was also a highly significant effect of Age on Number Relationships Combined Reaction Time ($F(4, 2381)= 14.946$; $p = 0.001$). Tamhane 2 post-hoc tests showed that there were no significant differences between 9- and 10-year-olds; 9- and 11-year-olds; or 10- and 11-year-olds. However, there were highly significant differences between 7 - and 8-year-olds, 7- and 9-year-olds; 7- and 10-year-olds, 7- and 11-year-olds, 8- and 10-year-olds and 8- and 11-year-olds. Surprisingly, the significant differences were in the direction of older children having longer reaction times.

## Discussion

As regards scores, there was a strong improvement with age on all groups of tests. For both Number Magnitude and Number Relationships, the biggest age differences were between 7- to 9-year-olds and 10- to 11-year-olds. For Number Meaning, the biggest age differences seem to be between 7-year-olds and older children, and between 9-year-olds and older children. This could, however, reflect a slight tendency for ceiling effects in the older children for Number Meaning. All test group scores correlated significantly with one another. Both Number Meaning and Number Magnitude scores were strong independent predictors of Number Relationships scores, suggesting that basic

numerical abilities and concepts predict more formal arithmetic, at least in a group with low mathematical attainment.

As regards *reaction times*, older children were faster than younger children at the Number Meaning and Number Magnitude tasks; but older children are slower than younger children at Number Relationships tasks. A related finding was that faster reaction times were related to higher scores for Number Magnitude and Number Meaning, but slower reaction times were related to higher scores for Number Relationships. These are somewhat puzzling results, but perhaps suggest that older and children are using more reflective, analytical strategies, that are more effective but slower, for this more formal group of tests. Further research could investigate what these strategies are, and whether these strategies are developed predominantly in ,response to instruction, or more spontaneously through experience.

Future research should, in particular, investigate how the different components develop longitudinally and respond to intervention. It would also be interesting to investigate how these components develop in typically achieving children, and the extent of any age differences in such a group.

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# Catch Up® Numeracy: An intervention for children struggling with numeracy

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**Keywords: mathematical difficulties; young children; intervention; arithmetic; number concepts**

## Background

Mathematical difficulties are very common. The prevalence of developmental dyscalculia has been estimated as approximately 6%, although there are many debates about the exact definition and nature of such a condition. In addition, many people, who would not be described as dyscalculic, have severe numeracy difficulties that persist into adulthood. For example, Parsons and Bynner (2005) reported that about 22% of a large adult British sample had numeracy difficulties severe enough to cause serious practical and economic problems.

Moreover, arithmetical ability is composed of many components, which do not always go together (Dowker, 2005; Gifford and Rockcliffe, 2012). For example, both among typical individuals and among patients with mathematical disabilities caused by brain injury, it is possible to be good at exact calculation and bad at estimation, or vice versa.

Dowker (2005) looked at unselected 5- to 9-year-old children's performance on arithmetical calculation tests, derived fact strategy use and computational estimation. These correlated significantly but not perfectly. For example, a strong independent relationship was found between derived fact strategy use and estimation, even after measures of arithmetic performance were accounted for. But it was possible to find children with marked dissociations in either direction between any two of these skills.

The components of arithmetical ability can be grouped into several categories (e.g. procedural, factual, conceptual), but each of the categories has numerous subcomponents. Levels of performance in different subcomponents are influenced by a multitude of interacting factors: brain-based (both innate and acquired), social, cultural and educational. Any statement that arithmetical ability is purely the product of a single factor is over-simplified. No two children (or adults) with arithmetical difficulties are the same (Dowker, 2005; Dowker and Sigley, 2010). It is important to find out what specific strengths and weaknesses an individual child has and to investigate particular misconceptions and incorrect strategies the child has.

Some interventions, targeted at children with severe difficulties, are very intensive, such as Mathematics Recovery, or Numbers Count. Half an hour per day of individualised intervention is typical in such projects. At least 15 to 20% of children experience significant mathematical difficulties (Dowker and Sigley, 2010), for many very intensive intervention may be neither practical nor cost-effective. Less intensive, but still targeted, interventions are needed for this group.

## Programme

The Catch Up® Numeracy intervention was developed by Catch Up®[1] in collaboration with Ann Dowker of Oxford University (Dowker and Sigley, 2010; Holmes and Dowker, 2013). The project was based on the 'Numeracy Recovery' scheme piloted in Oxford from 1998-2004 (Dowker, 2005).

The target pupils for the intervention have mostly been pupils in Years 2 to 6 who have numeracy difficulties, although it is also used with pupils in Years 7 to 9. Schools select the children who are likely to benefit; but the main target group is children with moderate mathematical weaknesses (lowest 15-20%) rather than those with severe mathematical difficulties.

Children in the project receive interventions from trained teachers or teaching assistants during two 15 minute sessions per week. Each child is assessed individually by a trained teacher/teaching assistant using the *Catch Up Numeracy assessments for learning* in order to complete the *Catch Up Numeracy learner profile*. This personalised profile is used to determine the entry level for each of ten Catch Up Numeracy components and the appropriate focus for numeracy teaching. Children are provided with mathematical games and tasks targeted at their specific level in particular components.

The components addressed in Catch Up® Numeracy are not regarded as an all-inclusive list of components of arithmetic, either from a mathematical or educational point of view. They were chosen because research and discussions with teachers indicated their importance:

(1) Counting verbally (subcomponents: counting verbally from 0 or 1; counting on from a given number; counting back from a given number).

(2) Counting objects (subcomponents: counting objects; order irrelevance; repeated addition of objects; repeated subtraction of objects)

(3) Reading and writing numbers (subcomponents: reading numerals; reading number words; writing numerals).

(4) Hundreds, tens and units (subcomponents: number comparison; adding tens and units; subtracting tens and units).

(5) Ordinal numbers. The ordinal numbers assessment involves showing children pictures of bead strings and telling them that the bead with the dot in the middle is the first (or in some pictures tenth) bead. One bead in each picture is shaded, and the children are asked to say which bead it is (e.g. second; fourth; ninth; etc.).

(6) Translation (subcomponents: translating from objects to numerals; translating from numerals to objects; translating from number words to numerals; translating from number words to objects).

(7) Word problems.

(8) Derived fact strategies (deriving and predicting unknown arithmetical facts from known facts) by using arithmetical principles such as commutativity or inverse operations.

(9) Numerical and arithmetical estimation.

(10) Remembered number facts.

Each 15-minute teaching session includes: (1) a review and introduction to remind the child of what was achieved in the previous session and to outline the focus of the current session; (2) a numeracy task; and (3) a linked recording activity where

[1] Catch Up® is the working name of The Caxton Trust, a not-for-profit charity registered in England and Wales (1072425) and Scotland (SC047557) as well as a company limited by guarantee (03476510). Catch Up is a registered trademark.

the child records the results of the task in oral, written and/or concrete fashion, and where the child receives focused teaching related to their performance on the task and any observed errors. Children usually stay in the intervention for one term, although teachers may adjust the length of intervention in the light of each child's progress.

The resources used in the intervention are those that would be found in a primary school classroom, and the teachers or teaching assistants delivering the intervention are encouraged to use resources that the pupil is already familiar with from their daily mathematics lessons. For example, if the class lessons are covering shape and space, then work on the 'counting component' could look at counting specific shapes such as triangles or cubes. The most distinctive feature of Catch Up® Numeracy is not any particular task, activity or piece of equipment, but an approach based on targeting specific weaknesses.

## Evaluation

The participants in the first evaluation (Dowker and Sigley, 2010; Holmes and Dowker, 2013) were 493 primary school children (mean age 8 years 8.49 months; s.d.14.63 months). 395 of these received the Catch Up® intervention program. 50 were given the same amount of time for individualised mathematics work, which usually involved reviewing work done in the school lessons and was not specifically targeted at diagnosed individual strengths and weaknesses. 48 children received no intervention, except for their usual school instruction. They were all given Gillham and Hesse's (2008) Number Screening Test before and after the intervention. The mean chronological age and mathematical age of the subgroups are summarised in table 1.

| **intervention** | **Catch up®** | **Matched time** | **No intervention** |
|---|---|---|---|
| **number of children** | 395 | 50 | 48 |
| **Mean age (s.d.)** | 103.44m (13.25m) | 107.83m (13.21m) | 103.65m (17.7m) |
| **Mean mathematical age (s.d.)** | 96.89m (16.65m) | 95.98m (14.7m) | 93.91m (15.52m) |

Table 1 mean age and mean mathematical age of the three subgroups

ANOVAs by subgroup revealed no significant initial group differences in chronological age, or mathematical age.

## Results

The ratio gain (months gained in mathematics age divided by the number of months between initial and final testing) was calculated. The 395 children who received Catch Up® intervention showed a mean ratio gain of 2.87 (s.d. 2.89); i.e. on average they made more than twice as much progress as would be expected from passage of time alone. The 50 children who received matched time intervention showed a mean ratio gain of 1.47 (s.d. 1.78). The 48 children who received no intervention showed a mean ratio gain of 0.86 (s.d. 3.17).

ANOVA with subgroup as the factor and ratio gain as the dependent variable, showed a very significant effect of subgroup ($F(2, 490) = 14.67$; $p < 0.001$. Tamhane 2 post-hoc tests showed that the Catch Up® intervention group made significantly higher ratio gains than either of the other groups – whose gains did not differ from each other.

Thus, the children who received the Catch Up® intervention made more than twice as much progress as might be expected from the passage of time alone and significantly more than the other subgroups.

More recently, a research project was funded by the Education Endowment Foundation (Sharples et al., 2015). This evaluation was a randomised control trial

(RCT), involving 324 pupils in 54 primary schools, with 108 Teaching Assistants (TAs). Pupils were randomly assigned to Catch Up® and control groups, with equal numbers in each group. The findings again showed a significant difference between those who received Catch Up® Numeracy, and those who received only the standard school-based help.

**Discussion and implications for future research**

The external evaluators concluded that Catch Up® Numeracy led to a 'noticeable improvement in numeracy outcomes' and that 'one to one teaching with Teaching Assistants is an effective strategy to increase numeracy skills in Year 2-6 pupils' (NFER, 2014: 5). Another RCT study is still ongoing to compare Catch Up® Numeracy intervention with matched-time one-to-one teaching. Current research is also looking at whether intervention in specific components influences progress in other components. Further research should also investigate the long-term effects of the intervention and, for example, whether it influences performance in GCSE and A level.

Catch Up® Numeracy is listed as an evidence-based intervention to support TAs in their one-to-one instruction, in the EEF (2015) *Making Best Use of Teaching Assistants Guidance Report* (Sharples, Webster and Blatchford, 2015). It is also included in the European Platform for Investing in Children (EPIC)'s list of evidence-based practices.

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# Is there a 'crisis of statistics'? A discussion and some suggestions for teaching

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**Keywords: statistics; representativeness; causation; big data; data analytics; identity; social media; globalisation; technology companies**

## Introduction

During most of our lifetimes, it has been accepted that, in most countries with a developed civil society, citizens and policy makers could rely on statistics produced by government or other agencies, to set a baseline for discussion and decision making. Thus we quote results from the European Labour Force Survey, or figures compiled by the World Bank, in discussions of unemployment. Or we use national household surveys, to estimate the numbers of victims of crime, or of those who are chronically ill. We have been able to take good statistics for granted.

> Recently, however, these previously accepted resources are facing new challenges. For example, a recent article in *The Guardian*, "Have statistics lost their power in public policy discussions?", has raised questions regarding the role of statistics in public discussions: "Antipathy to statistics has become one of the hallmarks of the populist right, with statisticians and economists chief among the various 'experts' ostensibly rejected by voters in 2016." (Davies, 2017).

I first briefly consider key points of the statistical approach. I explore the idea of a 'crisis' in statistics, and how it may be exacerbated by trends such as the growth of 'identity politics', developments in social media, and seemingly inexorable globalisation processes.

Recently, 'big data' and data analytics have seemed to offer new solutions to contemporary problems. I consider the potential, and the limitations, of these approaches. One problem, now clearly serious, is their tendency to be taken over by certain interests that are being found to threaten individual privacy, and even possibly to undermine rational, open, public discussion of values and policy.

In this paper, I reaffirm the value of statistics as a basis for policy, and consider ways that its ideas can be brought to life in the adult mathematics classroom.

## The statistical approach

We need to understand that 'statistics' can refer to three different aspects of the discourse, though of course they inter-relate: (i) statistical *data*: (ii) statistical *techniques* of data analysis (e.g. averages, measures of spread and correlation, statistical models); (iii) the *discipline*, including 'experts' and professional standards.

Key examples of the ideas and procedures of statistics include:

(a) The *representativeness* of the sample of data available, relative to the specific population about which one wishes to draw conclusions, and familiarity with the methods of representative sampling and with the drawing of inferences from samples;

(b) The importance of comparable *measurements* across all members of the sample, and knowledge of ways to assure the quality of such measures; and

(c) The difference between dependable *causal explanations* for what is observed and (possibly accidental) correlations, and ways to allow for this ambiguity so as to design informative studies that build dependable knowledge.

From the late 17th and 18th centuries, the idea gained ground in Europe that statistics should be used to understand an entire population (not only potential soldiers, or tax-payers). In the 19th and 20th centuries, specific indicators, clearly defined and systematically produced, were constructed for simplifying description of diverse and complex populations. For example: population size; classifications of disease; national income statistics (e.g. GDP). Surveys and opinion polls of representative samples of the population (and of subgroups), using random sampling methods were introduced. Experimental designs (nowadays called randomised controlled trials - RCTs) were introduced for agricultural trials and extended into the study of medicine and psychology. (For further details, see Evans, 2018.)

## Is there a crisis of statistics?

Some of the dimensions of the current situation include: an increasing lack of trust in statistical data, and a consequent decline in their authority. For various reasons this has become particularly evident in the UK and the USA over recent years. This leads people to brand any evidence that seems contradictory to their preferred worldview as 'fake news', or as something made up by 'experts'. Thus there are trends towards a lack of generally accepted baselines for discussing competing claims about society; and consequently a resort to 'speaking one's own truth', and drawing on 'intuition' and emotion as alternative bases of knowledge.

We can consider further some aspects of these reactions to statistics. The need to govern the population as a whole conflicts with (increasing) pressures to respond to feelings of particular citizens in a particular place and time. This can lead for example to a mismatch between what politicians say about the general state of the labour market, and local experience of it. Such problems are aggravated by several trends: (a) the fragmentation of identities (e.g. gender) and the emphasis of difference; (b) the reshaping of global economy, making definitions more complex e.g. of GDP; (c) the difficulty of measuring, say *intensity* of employment, or *commitment* to actually exercising one's 'voting preference' on election day; and (d) strains on comparability across time, and comparability across nations, for example with PISA scores.

Good quality data need to be produced with care, requiring time. So statistics are often 'out-of-date', by the time they appear!

## Here comes Big Data

Recently Big Data and data analytics have seemed to offer new solutions to problems resulting from an essential lack of certainty surrounding efforts to understand society, a limited research capability, and the need to make timely decisions in rapidly changing conditions. Big Data can be characterised as the availability of exceedingly large amounts of data, harvested in an ongoing way, as a by-product of other processes, and requiring the extensive use of electronic technology for capture, analysis, and presentation.

Examples of *Big Data* include the use of loyalty cards, allowing monitoring of purchasing behaviour, plus correlation of such data with a number of demographic variables, in order to target marketing, plus an option of experimenting with the effects of differential 'special offers'. A further example is the 'harvesting' of electronic texts, from individual acts of communication, which earlier might have been assumed to be

private, e.g. information searches, social media posts. These texts can now be subjected to techniques including *data mining*, where many of the data analysis decisions are made by *algorithms* run by machines, rather than by human analysts. These are supplemented by *data linkage* (of data on a person from several databases), and *sentiment analysis*, used to striking effect by certain companies in the US election and the UK referendum (Cadwalladr, 2017-18).

## Issues with Big Data and Data Analytics

In methodological terms, the data involved is 'big' indeed, but there are several serious limitations (Evans, Ruane & Southall, 2019). First, the approach often involves 'haphazard' harvesting of large amounts of data – indeed impressive amounts. However, a huge sample can still be biased – see the example of the contrasting predictions made in the 1932 US Presidential election (Marsh, 1979). If there is no known sampling design, generalisation *to any recognisable population* will not be reliable.

In many cases too, the data comes without settled categories, since people can take on self-selected identities. This means that data from one database may be hard to link reliably with data from another. Further, even if you have access to a huge data set, *correlation is – still – not causation!* Data analytics is 'suited to detecting trends, sensing the mood, spotting things bubbling up' (Davies, 2017), not to developing widely applicable theories, with any lasting validity.

Other more political issues arise for the responsible citizen. These relate to the authenticity of the 'informed consent' offered to potential users of data harvesting software, and the ownership of the data given. Data linkage raises not only technical issues (about accuracy), but also issues of personal privacy: Would you want data from your medical records to be linked to your income tax return information, or to your Facebook page? For a prescient picture of a possible future, see the striking novel, *The Circle* (Eggers, 2014).

The providers of all this data may never know what the data says about them – much less how it might be interpreted later by an unknown, suspicious, user.

## A covert crisis?

There is thus a second, 'covert', crisis of statistics, based on opposing ideas of knowledge. On the one side, we have the 'experts' of the Office of National Statistics – bound by research ethics, and monitored by the UK Statistics Authority – and on the other, the experts of Google, Facebook, other less known actors, such as Cambridge Analytica (Cadwalladr, 2017-18), and their various customers. These latter appropriate data from individuals, link it with information available from public or privatised databases, and sell it on to a range of customers, to be used for purposes such as 'tailored messaging', by marketers, politicians, 'opinion formers'. Some of these interests are oriented to maximising advertising revenues and to maximising the appropriation of other people's data, as indicated here – the Corporations. Others may be oriented to undermining rational, open, public discussion of values and policy – the Ideologues. There are currently (April 2018) official investigations ongoing into the way both groups used 'big data' methods by the Brexit campaigns in the UK, and by the Trump campaign in the USA in 2016.

Without statistics, and social research more generally, made available publicly and discussed freely, we cannot construct unambiguous, objective, potentially consensus-forming claims about society – nor can we provide a corrective to faulty

claims. In such a situation, there will be few mechanisms to prevent people from instinctive reactions and emotional prejudices.

Thus, the battle is not between 'an elite-led politics of facts versus a populist politics of feeling' (Davies, 2017). Rather, it is between those committed to public knowledge and argument versus those who profit from the privatisation of information and the 'ongoing disintegration' of public knowledge and argument.

## Conclusion: What might be done?

As citizens, it is important to rethink our relationship with companies in IT's 'Fearsome Five': Facebook, Apple, Amazon, Microsoft, Google. (Here the reader may wish to think which companies control most aspects of their lives.). There is much discussion currently in the press about what might be done (see also Evans 2018).

As teachers of adult mathematics/numeracy, we can encourage our students to reflect on their positions with respect to the trends described above, using surveys that can be done in the classroom, focussing on the variations in their uses of social media technologies. This could include all phases of social survey design, including: what to research and how to conceptualise e.g., say, a 'zombie facebook user'; how to deal with issues of sampling, measurement, and explanation; how to interpret data, especially anomalies, and how the results might be *used*.

In addition, many countries have available on the web a wealth of statistics produced by government or other agencies. In some cases, these have been made available in convenient form to generate discussion. In particular, the work of Hans Rosling and his associates has aimed to make available a range of statistics in the form of dynamic interactive graphics that can challenge mistaken preconceptions about the world, at the same time as assuring us that 'things are better than you think'. This work represents, in my view, a ground-breaking example of the use of statistics for the support of knowledge development, and for focussing on insightful and collective ways of seeing our way to humane solutions (Rosling, 2018).

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# Royal Institution Primary Mathematics Masterclass Programme: Launch of "Off the Shelf" (OTS) Masterclasses

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*The Royal Institution (Ri)*

**Keywords: mathematics enrichment; outreach; key stage 2; extra-curricular; inspirational**

## Masterclasses which are ready to use: perfect for the busy teacher or presenter. Just add enthusiasm!

The Ri Mission is to create opportunities for everyone to discover, discuss and critically examine science and how it shapes the world around us.

Thousands of children have engaged with Ri Masterclasses over the years, but there is capacity for thousands more to do so. These new resources will enable enthusiastic teachers to bring Masterclasses to their local schools with a minimum of fuss and additional time investment. A Primary Masterclass Series brings together children (normally in year 5) from a group of local schools to attend a series of (usually) six sessions, on a wide variety of inspirational topics from beyond the primary curriculum. The mathematics might be pure or applied or a mixture. Ideally the children will encounter a range of presenters, although this is not always possible.

These new OTS Masterclasses include a PowerPoint presentation, notes for the presenter and helpers, worksheets and a full list of any other necessary resources. There is a range of topics so that a series organiser can ensure there is breadth as well as depth for the students. Some of the topics are drawn from our previously published book of Masterclass ideas (Ri, n.d.). Eventually there will be more than 20 different topics to choose from.

Full support for anyone wishing to organise a Masterclass series is available, as usual, from our dedicated Masterclass team based in London. This support includes but is not limited to: training for presenters; templates for all the necessary paperwork such as invitations, letters, risk assessments etc.; certificates for all attendees completing the series; mathematical support via our "ask the Ri" feature; invitations to celebration events and other follow up events in London or at regional meetings.

We support a network of Masterclasses all over the country but we would like many more children to benefit from this opportunity. OTS Masterclasses are one way to achieve this goal. To get involved or ask any questions please email masterclasses@ri.ac.uk .

## Reference

Ri (n.d.). *Mathematics Masterclasses: Resources for the Primary Classroom* http://www.rigb.org/education/masterclasses/masterclass-resources

# The Culturally Responsive mathematical Mathematics Teacher Educator: Exploring cultural responses to a symbol and implications on pedagogy

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**Keywords: culture; cultural responsivity; ethnomathematics; symbol; semiotic**

Culturally relevant pedagogy uses the cultural knowledge, prior experiences, frames of reference and learning styles of culturally diverse groups of students to make learning more relevant and effective (Gay, 2009) and to strengthen the connection between learning within and outside of the school environment (Weist, 2001). When incorporated effectively it validates students' cultural background, their ethnic history and their societal interests and addresses their socio-emotional needs (Orey and Rosa, 2013).

The universality of mathematics has long been accepted. Mathematical activities such as counting, locating and measuring are applicable in the non-human universe and are therefore not creations of human beings. There is a clear argument that 10 plus 10 always was and still is 20 no matter what era or context one finds oneself and no matter what language or symbol one uses: 'It is not time, space or culture dependent' (Arda Cimen, 2014:525).

Such acceptance contributes towards a possible misconception that sense making in mathematics is therefore also based on the idea of cultural independency and universality; that it is culture-free knowledge. Early mathematics relies heavily on the use of abstract symbols that serve as metaphors of concrete ideas. The importance for young learners to make connections between the abstract representation and the real experience is well established (Francois, 2010; Boaler, 2015). But rarely do we see teachers incorporate learners' values, traditions, beliefs, language and habits reflective of their own culture into the daily mathematics lesson.

Ethnomathematicians assert the cultural relativity of mathematics and bemoan the eurocentrism of mathematics education claiming it pays little regard to the socio-cultural differences of the learners that it serves (D'Ambrosio, 1985; Weist, 2001; Ascher, 2002; Orey and Rosa, 2013). Assertion that cultural difference is relevant to the teaching and learning of mathematics is a key idea that this small study pursues.

## The Study

Data was generated from an activity that exposed particular attachments to a numerical symbol. With prompts (holidays, domestic chores, the classroom and shopping) 51, culturally diverse student teachers produced personal mind maps of the symbol '40', which without any attachment is an unmotivated symbol made up of purely three straight lines and an ellipse. They produced a total of 695 attachments within the 51 samples that were then scrutinised through documentary analysis, volunteer interpretation and analysis of recorded group discussion. Figure 1 provides a basic framework to consider degrees of cultural responsivity and the associated level of adjustment in Initial Teacher Education (ITE) mathematics module design.

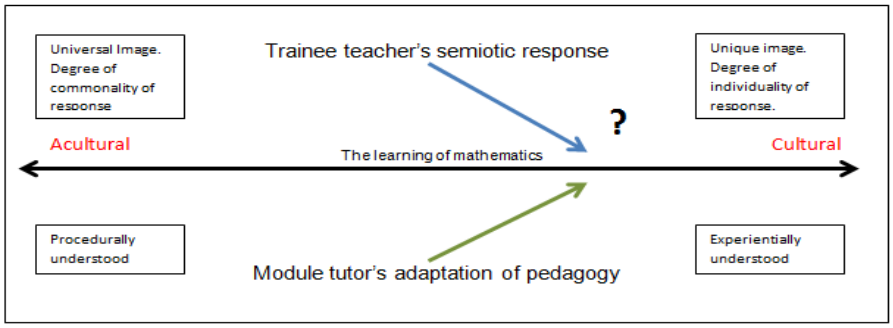



Figure 1 Framework for cultural responsivity in ITE provision

The three themes that emerged from a non-predetermined coding process of the mindmaps gave valuable evidence of the presence of a cultural response. The first theme, 'A moment in time' (Figure 2) considered how many of the attachments to the numerical symbol were snapshots of the present and would in fact not be related to the symbol in the near future or in the recent past. Such attachments as 'the cost of a pair of jeans from Top Shop', 'my mobile phone contract', 'the tax I pay on my car', and 'my Mum is 40' captured moments in time that were mostly related to money and age. This categorisation was not effective in locating a group cultural response but highlighted the personal nature, the individuality of the semiotic response. It highlighted the unpredictability of the response for the educator and supported the notion that mathematics is socially constructed, influenced by who, when and why and therefore not built on universal truth. This categorisation relates strongly to Borba's (1997:267) explanation of culture as the relationship between a person and their world 'in a historical moment' and Gay's (2009:197) definition of culture as 'an everyday, all the time phenomenon'.

Another emergent theme was 'student/tutor alignment'. By comparing the tutor's own attachments to 40 with those of the students' it emerged that the students' attachments were not culturally relevant to the tutor's life outside the mathematics classroom. Unlike the students, the tutor did not relate '40' to phone battery life, calorie counting, gym membership, vodka alcohol percentage, budget for a weekly shop or the price of a small tattoo. Nor did the tutor attach 40 to words such as 'old' or 'middle aged' yet these connections were made repeatedly across the documents. With a 30 year difference between the modal age of the sample and the tutor's age and the fact that all are full time students, it was likely that the disparity was a result of generational and economic difference between student and tutor. Such a display of group cultural response highlighted how the many real life scenarios presented within lessons may be irrelevant to the students' real lives. From this data the importance of consideration of the tutor's own cultural identity and where they fit as a member within the various cultural groups in the classroom was established. Barta and Brenner (2009) emphasise this consideration, warning that teachers who fail to consider their own cultural identities run the risk of not accepting and respecting the diverse cultures displayed by their learners.

The third emergent theme was religious reference (figure 3) and again these findings evidenced significant group cultural responses. The number 40 has great symbolic meaning and the connection to religion in the gathered data was interesting in relation to cultural responsivity. It possibly highlighted the many missed rich opportunities to work from students' own cultural perspective in mathematics. References to Lent in Christianity, the number of days and nights Moses spent on Mount Sinai in Judaism and the age of 'best wisdom' in Islam appeared amongst several other religious references.

Race (2011) and D'Ambrosio (2001) warn of educators risking tokenism in culturally responsive pedagogy. Female students amongst the sample who wear the hijab and whose mere appearance declares a defined cultural group may influence the teacher's cultural response but such commonality in this case masked a deeper cultural affiliation. The occurrence of the Islamic attachments to 40 was hugely overshadowed by the prevalence of the students' generational cultural responses such as mobile phones, shopping, fast food and fitness classes. An Islamic Tile Tessellation activity is valid for mathematical curiosity but teachers may need to be wary of its superficial cultural content.

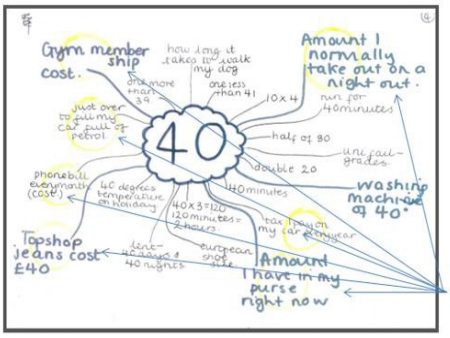



Figure 2 A moment in time

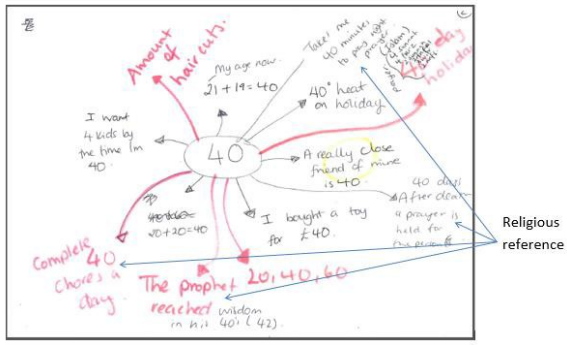



Figure 3 Religious reference

A more detached approach to determining cultural response was to use 24 volunteers unknown to the research sample to attempt to determine the age, gender and ethnicity of the authors of the anonymized documents. This produced 90% accuracy in determining the gender of the author and 100% accuracy in determining which respondents were male. Analysis of the documents' for the authors' age produced an accuracy rating of 74% and for ethnicity an accuracy of 73%.

Due to the semiotic complexities of aligning the intended meaning by the author with the interpreted meaning by the reader, authors and interpreters were brought together to attempt to connect the mathematical scenarios depicted in the images of 40 with the human story that had led to them. Analysis of recorded group interviews and accompanying notes gave insight to inherent meanings. Some documents were predominantly universal responses, rigid, formal and non-contextual – 'double 20', '5 × 8', and 'an even number'. Others were rich with links to worlds beyond the universal: they were fluid images where one connection led to another and stories could be told. In interview, it emerged that the difference between the documents was the cultural framing of mathematics during their own schooling.

The findings from the data analysis indicate the presence of a cultural response to a mathematical symbol and support D'Ambrosio's (1985) conviction about the value of explaining and understanding mathematics from diverse cultural perspectives.

## Impact on ITE mathematics module design

Changes have since been made to Initial Teacher Education (ITE) mathematics modules that have begun to impact on the way our student teachers are approaching early mathematics pedagogy. The incorporation of the 40 activity early in the mathematics programme generates a discussion on how mathematics provides starting points for valuing diversity in the classroom. Mathematics session objectives refer to Teachers' Standards such as *'to respect the social, cultural, linguistic, religious and ethnic backgrounds of children'* and '*to be committed to raising the educational achievements*

*of all groups'*. Consideration of the historical influence of women in mathematics, the dominance of the eurocentric perspective, how the media can influence a nation's attitude and parental influence and involvement in the learning process are points built into the programme for discussion and investigation.

The primary mathematics classroom is modelled through activities such as creating mathematical story problems reworded with the incorporation of familiar items and family names that the students provide in a biographical questionnaire and through students bringing their own mathematical problems encountered in the home to be solved in the classroom. Students are requested to bring photos from home that are used to pose 'where's the mathematics', and 9am mathematics lessons begin with 'what mathematics have you already done today?'. Grouping within University sessions are carefully structured to encourage cooperative interactive working using heterogenous grouping and students are supported to do this when lesson planning for school.

The ITE programme encourages student teachers to see mathematics as a human endeavour performed by real people serving real needs and interests. By getting to know the lives of the student teachers, a child-centred approach can be modelled that supports them to become culturally responsive mathematics teachers. Having been given the tools and skills to teach a multicultural mathematics curriculum, tutors observe student teachers in mathematics lessons making reference to their pupils' lives outside the classroom, using relevant real life problem solving activities that validate children's cultural background and their societal interests.

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# Mapping Diagrams: Four Sessions, One Theme. Using Mapping Diagrams in Algebra, Calculus, and Complex Analysis

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**Keywords: mapping diagram; function; equation; linear; quadratic; cubic; calculus; integration; complex analysis; visualisation**

## Introduction

Mapping diagrams were the central theme for four sessions presented at BCME9. A mapping diagram is a visual tool for understanding functions and equations that provides clarity for several procedures and concepts that are not easily treated with a Cartesian graph. The argument (input) of a function - an element of the domain (source) set – is visualised as a point on a number line (axis). Separately the corresponding value of the function (output) - an element of the codomain (target) set is visualised on a second parallel axis. An arrow from the input to the output visualises the relation between the two numbers, see figure 1.

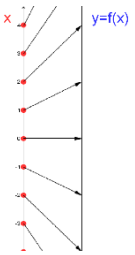



Figure 1.
Mapping Diagram

A mapping diagram can display many arrows comparable to a table, or the point in the domain can be moved with graphic technology giving a dynamic visualisation of the function. A similar mapping diagram can be created for functions of complex numbers by replacing the two axes with two planes visualising the complex domain and co-domain, see figure 2.

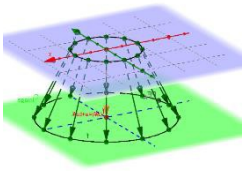


Figure 2.
Mapping Diagram for Complex Function

The mapping diagrams in the figures and used in the four sessions were created using GeoGebra. The content of each session will be discussed separately in what follows. Links to the GeoGebra Books for all the sessions, references, and many other related materials can be found at the website: users.humboldt.edu/flashman/Presentations/BCME/BCME.LINKS.html.

## Visualising Quadratic, Cubic, and Quartic Equation Solutions: An Introduction to Complex Numbers, Functions, and Mapping Diagrams

Solving equations is one of the early and dominant tasks presented to students as they begin the study of algebra. Starting with linear equations, students learn to solve the general quadratic equation and some more accessible cubic and quartic equations with little or no visualisation provided. A visual approach is less available when complex numbers are introduced to solve even the simple quadratic equation $x^2 + 1 = 0$. In this session mapping diagrams of functions are used in two ways: first to visualise the steps in the algebraic solution of linear and quadratic equations by recognising the functions in these equations as compositions and then to visualise the nature of complex solutions to quadratic, cubic, and quartic equations.

For example solving the linear equation $3x + 1 = 5$ is connected to the composite mapping diagrams for the functions $f(x) = 2x$ and $g(x) = x + 1$. The

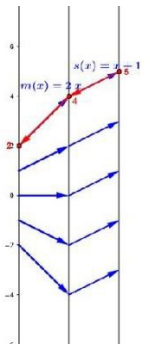


Figure 3. Mapping Diagram Solving Linear Equation

number 5 is represented by a point on the final target axis, and the algebraic steps are visualised by reversing the direction of the arrows (inverting the functions) of the composition to arrive at the point/number solution, $x = 2$ on the first source axis, see figure 3.

A quadratic equation represented in the "vertex" form, for example, $2(x-1)^2+1=9$ also is connected to a mapping diagram for a composite of four core functions and the algebraic steps for solving the equations are visualised similarly when real roots exist (see figure 4) and later even for complex roots using complex mapping diagrams.

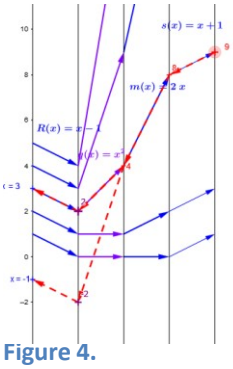


Figure 4. Mapping Diagram Solving Quadratic Equation

Cubic, and quartic equations are not solved algebraically in the school curriculum but can also exhibit complex number roots. For these equations, mapping diagrams for complex functions can visualise dynamically important connections between the roots and the coefficients of the polynomial functions. For example the following mapping diagrams (figures 5 and 6) visualise the cubic equation $x^3-1=0$ has one real and two complex conjugate roots, and the equation $x^4+1=0$ has two pairs of complex conjugate roots.

Figure 5. Mapping Diagram Solving a Cubic Equation

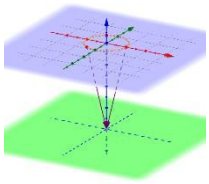
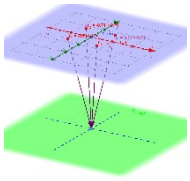


Figure 6. Mapping Diagram Solving a Quartic Equation



## How Many Ways Can You Solve a Quadratic Equation Visually? From the Greeks to 21st Century Technology

Quadratic equations have a long history, going back at least to the Greeks and Euclid's Elements. The visual solution of these equations starts with finding the defining segment (root) for a square with the same area as a given rectangle and the Pythagorean Theorem for finding the root for the sum of two squares. Quadratic equations are visualised as an unknown square with an attached rectangle being equal in area to a given square. The ancient solution of this problem used the completion of the square in a literal sense, see figure 7.

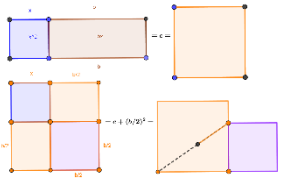
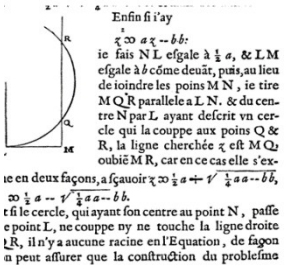


Figure 7. Completing the Square: Quadratic

Later Descartes revolutionised the problem by treating the problem as one about lengths of segments where the product of segments is visualised by another segment, see figure 8. The solution of a quadratic equation is one of the first uses Descartes makes of this visualisation, solving the problem without any explicit mention of geometric squares - though still completing the square with a construction similar to that of the ancients.



Enfin ſi i'ay
$z \infty az - bb$:
ie fais NL eſgale à $\frac{1}{2}a$, & LM eſgale à $b$ cõme deuãt, puis, au lieu de ioindre les poins MN, ie tire MQR parallele a LN. & du centre N par L ayant deſcrit vn cercle qui la couppe aux poins Q & R, la ligne cherchée $z$ eſt MQ; oubiẽ MR, car en ce cas elle s'exprime en deux façons, a ſçauoir $z \infty \frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$, $z \infty \frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb}$.
Et ſi le cercle, qui ayant ſon centre au point N, paſſe par le point L, ne couppe ny ne touche la ligne droite QR, il n'y a aucune racine en l'Equation, de façon qu'on peut aſſurer que la conſtruction du probleſme

Figure 8. Descartes Solution of a Quadratic Equation

The use of the graph of a quadratic function to determine the axis of symmetry by shifting the graph visualises the process of completing the square, but does not

show the actual algebraic steps for solving the equation after that step. The algebraic steps are visualised by the mapping diagrams as treated above in the section on solving equations.

**Making Sense of Integration Visually: Mapping Diagrams for Calculus**

The tradition of the 20th century was to link the definite integral with finding the area of a region in the plane bounded by the graph of a function on a compact interval. An alternative treatment connects the definite integral to solving a differential equation and the net change in any solution's value for a compact interval. This second approach is closely connected to the dynamics of change and motion that motivated Napier in his definition of the logarithm and Newton in his understanding of fluxions. A visualisation based on this dynamics approach to the definite integral is nicely done by using mapping diagrams.

Initially the mapping diagram visualises linear functions as a magnification with a focus point, see figure 9. The magnification factor for the linear function that defines the derivative of a function is then the model for estimating the function for a small interval about a number of interest, $x=a$. This is the differential estimator, $\Delta y \approx dy = f'(a)\,dx$. This is also visualised with a mapping diagram showing that

$f(a+dx) \approx f(a) + f'(a)\,dx$, see figure 10.

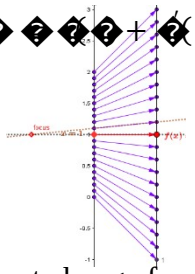


Figure 9.
Mapping Diagram
for $f(x) = 2x + 1$

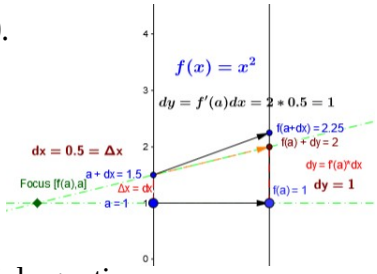



Figure 10.
Mapping Diagram
for differential:
$f(x)=x^2$, $x$=1,
$dx$=0.5

To estimate the net change for solving a differential equation $F'(x) = P(x)$ over the interval $[a, b]$, we use Euler's method with unit solution $y_0 = 0$. The estimated net change will be a sum of differentials $\sum F'(x_k)\Delta x = \sum P(x_k)\Delta x$. The accumulation of these differentials can be visualised with mapping diagrams, so the sum estimates the net change in the value of a solution to the differential equation, $F(b) - F(a)$, see figure 11. This difference is independent of the choice of the initial value.

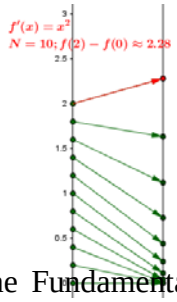



Figure 11.
Mapping Diagram:
Estimate of net change:
Accumulated Differentials

The limit of these sums defines the definite integral, and so the evaluation of the definite integral $\int_a^b P(x)\,dx$ is the net change in any solution to the differential equation, $F'(x) = P(x)$ *i.e.*, we have a definition of the definite integral and immediately one form of the Fundamental Theorem of Calculus for continuous functions: $\int_a^b P(x)\,dx = F(b) - F(a)$ with $F$ being any continuous function where $F'(x) = P(x)$ for all $x$ in $[a, b]$.

**Making Sense of Complex Analysis with Mapping Diagrams: A New Visualisation Tool Enhanced by Technology**

It is often said that the task of complex analysis is to generalise the calculus of real variables to a calculus for complex variables. The first difficulty encountered in complex analysis after visualising complex numbers by identifying them with points in a coordinate plane, with $x + yi$ corresponding to the point $(x, y)$, and making sense of

arithmetic for complex numbers is to visualise functions, linear and nonlinear elementary functions. Because the real calculus emphasises the graph of the function as the main visualisation, at this stage the difficulty of a graph for complex functions is primarily that of visualising a four dimensional Euclidean space where the graph of a complex function would appear as quadruple of real numbers- $(x, y, u, v)$: $h(x+iy) = u+iv$. By using mapping diagrams to visualise and understand real functions, the visualisation of complex functions can be handled comparably without needing "four dimensions".

Linear functions are the key to both differentiation and integration of complex functions as they were for real functions. Mapping diagrams for complex multiplication arise from an amplification from a focus point – the vertex of a cone and a twist-rotation resulting from the angle determined by the complex number multiplier, see figure 12.

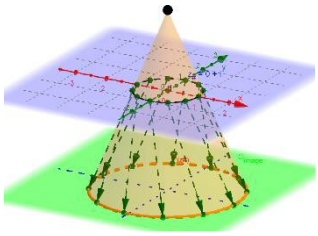


Figure 12. Mapping Diagram for $f(z) = (2+i)z$ applied to the circle $|z|=1$.

The magnification factor for the linear function that defines a complex function's derivative is the model for estimating that function for a small disc about a number of interest, $z = a + bi$, that is the differential estimator, $dw = f'(z)dz$. This is visualised with a complex function mapping diagram, see figure 13, $f(z) = z^2$, $z = i$, $dz = 1 + 0.5i$.

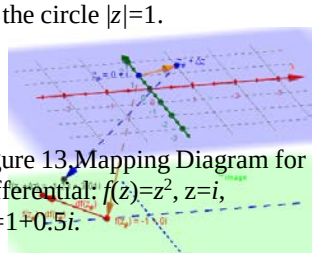


Figure 13.Mapping Diagram for Differential: $f(z)=z^2$, $z=i$, $dz=1+0.5i$.

The line integral for complex analysis is determined along a curve $\gamma$ in the complex plane given parametrically by estimating the net change for solving a differential equation $f'(z) = P(z)$ over the defining interval $[a, b]$ for the curve's parameters. As with the real integral we use Euler's method with initial condition $f(\gamma(a)) = 0$. The estimated net change will be a sum of complex differentials, $\sum f'(z_k)\Delta z = \sum P(z_k)\Delta z$. The accumulation of these differentials can be visualised with mapping diagrams, so the sum estimates the line integral over the curve, $\int_\gamma P(z)\,dz$, the net change, $f(\gamma(b)) - f(\gamma(a))$, where $f$ is a solution to the differential equation. The visualisations for the line integral can help see this accumulative limit, see figures 14 and 15.

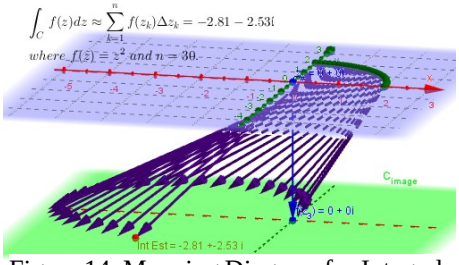



Figure 14. Mapping Diagram for Integral Estimate: $P(z)=z^2$ over circle $|z|=2$

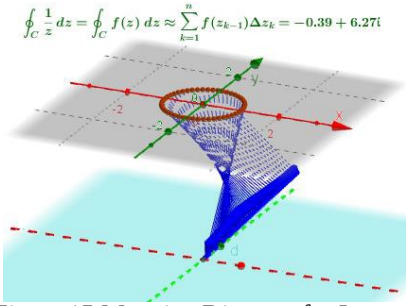



Figure 15.Mapping Diagram for Integral Estimate: $P(z)=\frac{1}{z}$ over $|z|=1$.

## Not the End. Only a Beginning

This meagre summary cannot capture the power of dynamic figures to visualise the concepts covered in the sessions, nor can it show other concepts that mapping diagrams support. These can be found in the materials linked initially in this summary or by the reader in their own future explorations.

# Inspiring Ideas for Maths Clubs

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**Keywords: Maths Clubs; Maths Camps; low-resource teaching; international collaboration; Kenya mathematics**

## Introduction

Experiences running Maths Camps in Kenya with the charity Supporting African Mathematics Initiatives (SAMI) have led to new ideas for running a Maths Club in a UK secondary school. This article aims to describe part of this journey, and highlight a set of inspiring, and ready-to-use resources for secondary school Maths Clubs. There is also a brief discussion on ways for secondary school Maths Clubs to collaborate with African partner schools, via online puzzles and investigations.

## About SAMI

SAMI is a mathematics educational charity, volunteer-led by teachers, lecturers, technology specialists and students. Its work covers a range of initiatives in several African countries including Kenya, Ethiopia, Ghana and Tanzania. The scope is diverse. The focus is improving educational outcomes from the primary school level up to Masters level, as well as community development initiatives - from farming to small businesses.

## Maths Camps

SAMI was established to support Maths Camps being run in Kenya by local educators alongside international volunteers. The aims were to show students that there is more to mathematics than what is taught in the school curriculum, to give school children the opportunity to do mathematics on a computer, and to give students an eye-opening experience of working with national and international teachers, university students and lecturers. These teachers, students and lecturers were referred to as ‘facilitators’ during the camp and they organised, coordinated and presented seminar and lecture material, while also leading workshops and other fun activities. Concepts and structures were introduced in an investigatory way, as opposed to teaching small areas of the curriculum. The camps in Kenya were so successful from their inception in 2011 that annual camps are also run in Ghana, Tanzania and Ethiopia.

The success of camps in other countries is very useful and important, it has helped SAMI to gain evidence that the model developed in Kenya works in new and distinct environments. Resources are shared between the camps and the key values are upheld and integrated at each one. While running Maths Camps in more countries is one measure of success, it is not scaling up the number of students who are able to access this sort of intervention. At present it remains a small scale project wherever it is run. Despite the Maths Camps being a very good model for learning, developing resources, and demonstrating that students can learn from interactions with computers and each other, the organisers believe that their Maths Clubs initiative, together with the Maths Camps, will better address the issue of scale.

## Maths Clubs

The Maths Clubs initiative was founded at the Kenyan Maths Camp. At the end of the Maths Camp, in addition to the software used, students are given a year's worth of resources for a weekly Maths Club, the *Maths Club Pack* (SAMI). All students who attend Maths Camp are encouraged to use the Maths Club Pack to establish or improve Maths Clubs in their own schools on their return. The Maths Club Pack (figure 1) consists of puzzles and tasks written up in both student and facilitator versions. Students are also given the opportunity to act as facilitators themselves in a session at the Maths Camp, so they can then take on this role back in their school.

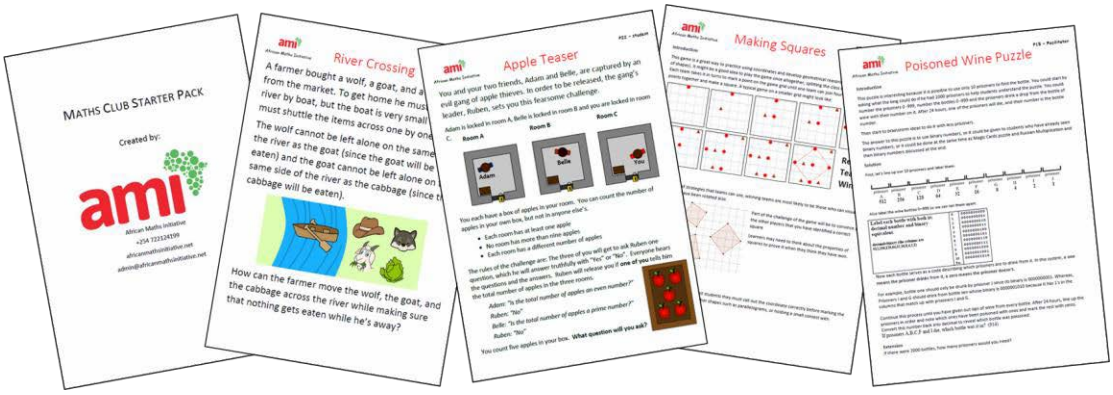



Figure 1. Example pages from the Maths Club Pack

Camp organisers undertake follow up visits to students in their schools, and support the students until they are able to run the clubs in an effective way. This initiative is not, however, limited to students who have attended one of the camps; many schools have been visited and benefitted from 'mini' Maths Camps. These are half or full day sessions that replicate the Kenya Maths Camp where they receive the resources and training to start a Maths Club. Maths clubs have also been an integral part of our Digital Communities Initiative, where we have used tablets to educate different sections of one community.

The Maths Clubs initiative is building a large community of mathematics educators, mathematics enthusiasts and mathematicians, providing an opportunity to empower students to take responsibility for their own learning process inside and outside of school. It is now developing into a sustainable, established programme that helps students not only in Kenya, but more widely in other parts of Africa and the UK. The initiative's ability to be easily adapted and integrated into different settings demonstrates its potential for being a highly scalable, low cost model that can be used in a wide variety of different learning environments.

## Maths Clubs in the UK

Although we have created these resources for a specific low resource environment in Kenya, they have turned out to be a very useful resource in the UK as well. This is an example of how designing educational material for a low resource environment can also be useful in a high resource environment (not so easy the other way round). By low-resource environment we do not necessarily mean no access to technology, but usually the schools have no access to the internet, and a low skilled-teacher to pupil ratio.

Two SAMI directors who are also teachers at the Lycée Français Charles de Gaulle in London have used their experiences in Kenya to establish a Maths Club in

their school. Students at the Lycee have enjoyed the puzzles in the Maths Club Pack, and have helped set up a website (SAMI) to record extra insights and new puzzles that they have worked on each week.

Experiences running this club have led to new ideas to add to the Maths Club Pack that is distributed in Kenya. We get our inspiration for resources from many websites e.g. Mathpickle, nrich, Numberphile, Wildmaths, and books e.g. Stewart (2009). Finding new puzzles for our Maths Club and the Maths Club Pack in this way takes time, access to the internet and a well-stocked library. It also requires expertise to choose good puzzles and to adapt them for the audience if necessary. While we build capacity through Maths Camps in Kenya, we are also continuing to build up resources for the Maths Club Pack from the UK. We have been fortunate to have been assisted by a student at Bath University who has created a set of ten new resources for the Maths Club Pack as part of a university module called Communicating Mathematics. She has used graphics to make the puzzles more engaging, and the solutions easier to understand for the students playing the role of facilitator in their schools.

## Collaborating with schools in Kenya

Another initiative being trialed at the Lycée Charles de Gaulle in London is to allow students from Kenya and the UK to work collaboratively online. Using the Lycee's website, we set a puzzle that is open to students for a couple of weeks (e.g. figure 2), and then students from both countries post comments on their thinking. Partner schools in Kenya without internet access are visited by the local team who will bring mobile tablets and internet hotspot for the students to access.

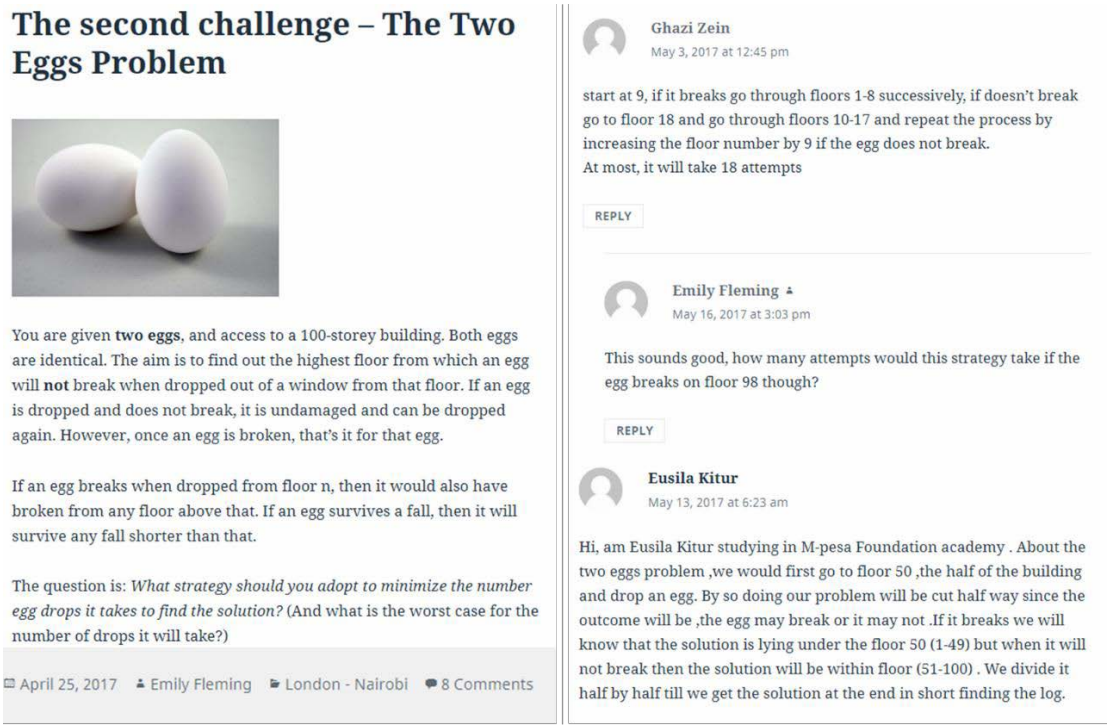



Figure 2. Screenshots from the Lycee website, showing one of the international puzzles and comments

We have tried this out with three puzzles in the last year and have more planned in the future. We hope to use these to help expand the minds of students in both countries.

## Discussion

The SAMI Maths Club Pack was initially established to provide more mathematical problem-solving opportunities to students in Kenya. Most problems are presented 'as-is', with simple accompanying visual aids and a set of facilitator notes to scaffold potential exploration and pedagogy. Evidence has been generated of engagement and enjoyment for the problems with groups of students in both Kenya and the UK. Building on from this initial implementation, we are now keen to explore more ways to build and add value to the network of students, teachers and mathematics enthusiasts that has formed through the Maths Clubs initiative. Advances in technology and the skills and dedication of volunteers in the UK and Africa have enabled many possibilities, such as resource-sharing and development of collaborative problem solving.

## Conclusion

Our experience with the SAMI charity establishing and running Maths Camps in Kenya for secondary school students helped us to develop mathematics materials and tasks that are fun and interactive, and that can be adapted for use in low resource school settings in Africa. Using the resources in the Maths Clubs Pack, this can be done independently by students acting as facilitators of Maths Clubs in their own schools, or with the help of Maths Club organisers and volunteers. Unlike the Maths Camps model, we believe the Maths Clubs model is very scalable, and it has already been adapted and successfully used in a high resource secondary school in the UK.

In the Maths Clubs model puzzles, games and other interactive and fun tasks are used to engage students in learning beyond simply what is required by the curriculum. The ideas and material we include in our Maths Clubs Packs are constantly changing, and we gain inspiration from many different sources including websites, books, games, and volunteers. More recently, our UK based Maths Club has evolved to include a dedicated website, and where students from the UK and Kenya can work collaboratively on Maths Club puzzles. While establishing and running a Maths Club does involve a commitment of time and expertise, we hope that we have convinced you that the Maths Club model is both easily adaptable and scalable, and encourage you to start one at your school.

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## Extension materials for Years 4-8

Tony Gardiner

*Independent*

**Keywords: primary to secondary mathematics; extension material; expectations**

It would be foolish to pretend that all students can achieve the same outcomes from school mathematics by the age of 16. But it does **not** follow that we must accept these *eventual* differences in achievement as being determined *ab initio*. Ultimate differences in inclination and performance cannot be eliminated, but it is clear (e.g. from comparing different countries) that outcomes are to a significant degree the result of adult attitudes and interventions along the way. In particular, teaching can make a difference! So we should look for ways of ensuring that **all** students have a chance to establish a solid grounding in elementary mathematics, and to teach for as long as possible *as if* each student may one day need to take their mathematics further than we or they expected.

From the late 1990s, official policy (in the UK) encouraged the precise opposite, presuming that some students should be encouraged to gallop on ahead, while others proceeded more slowly through standard material – even though the evidence showed that this approach failed both groups. The strategy would appear to have consolidated the long tail of underachievers in England (as compared with other countries), and it somehow also managed to sell short the 'top end' – until recently constituting half the international average at age 14-15: see, for example, the extract from table B5 (Bradshaw et al., 2010: 85) – bearing in mind that the table excludes "16 countries with scores below 430".

| | Below Level 1 | Level 1 | Level 2 … | Level 5 | Level 6 |
|---|---|---|---|---|---|
| England | 6.1% | 13.7% | 24.8% | 8.2% | 1.7% |
| Average | 8% | 14% | 22% | 9.6% | 3.1% |

Table1. Extract from Table B5 (Bradshaw et al., 2010: 85)

For more recent data:

(i) Table 1.5 in Sturman et al. (2012: 11) compares the performance of the Year 9 cohort in 2011 with the corresponding Year 5 cohort in 2007, and shows a statistically significant drop in performance

(ii) England's longer 'tail' is illustrated by the summary findings (2012: 23)

> In England, 7 per cent failed to reach the Low benchmark for either subject at Y5 and for science at Y9. A larger proportion of 12 per cent failed to reach the Low benchmark for mathematics at Y9. The equivalent figures for the highest performing participants were typically at or lower than 4 per cent for each subject at each age range.

(iii) There is also weaker performance at the 'top levels' (2012: 23)

> For Y5 mathematics and science and for Y9 science, England had between 40 and 50 per cent of pupils at the top two international benchmarks. For Y9 mathematics, the figure was lower at 32 per cent. The comparable figures for the highest performing Pacific Rim countries in each case were between about 60 and 80 per cent.

(iv) the general point is also illustrated by the comments of Jerrim and Shure (2016: 6):

> England's top achievers in mathematics do not stand out in the same way as our top scientists. […] the relatively poor mathematics skills of England's low-achieving pupils stands out as a weakness of England's education system. England's lowest achievers have mathematics skills that are significantly below the mathematics skills of the lowest achievers in several other countries. It is also notable how the bottom 10 per cent of mathematics performers in England trail those in Northern Ireland and Scotland, despite both countries having very similar average mathematics scores to England. Indeed, England has a particularly unequal distribution of 15-year-olds' mathematics achievement. The gap between the highest and lowest achieving pupils in mathematics in England is above the OECD average and is equivalent to over eight years of schooling.

In short, our attempts to lower expectations for those who find elementary mathematics challenging failed to improve outcomes; and individual acceleration for others may have led to short-term gains, but these evaporated as students faced the demands of secondary school.

Recognition of these outcomes has recently led to a rediscovery of the potential advantages of 'whole-class teaching', in which important ideas and methods are approached in a uniform, coherent way for all students. In its simplest form, *all* students initially pursue a common, simultaneous path through the curriculum. Hence differentiation has to be re-interpreted in terms of the expected *depth* of understanding, and the range of *variations* addressed, rather than simply in terms of the speed with which standard material is covered. This approach has many advantages. Students are expected, and given the necessary time, to make sense of important ideas and methods *while remaining within a common social unit*. Those who at first find the material easy are routinely challenged to understand basic material more robustly – in the spirit of the aphorism:

> If you eventually wish to build higher, you must first dig deeper.

Hence teachers need to develop a repertoire of *variations* designed to challenge all students to go beyond merely getting answers to simple problems, and to routinely extend those who appear to find the material relatively straightforward. In the early years of primary school such variations and additional challenges are best mediated through the teacher, holding the class together, and avoiding implicit or explicit discrimination (even if some students do regularly require additional support, and others consistently appear to be ahead of the game). However, at some point before the end of Year 6, there may be a place for extension materials beyond what most teachers are able to generate for themselves. The *1st, 2nd, 3rd Books of Mathsteasers* (Gardiner, 2018)) offer precisely such a resource.

These extension materials are directly linked to the *Maths – No problem!* series of primary textbooks for Years 4, 5 and 6 (Yeap, 2015, 2016). Each of the mainstream textbooks group the year's work into 14-15 chapters of varying lengths (3 to 30 lessons). The corresponding book of *Mathsteasers* follows exactly the same sequence of chapters, supplementing each with 10 to 20 problems, which vary and extend the work of the relevant chapter. Structured answers/solutions are given at the end of each chapter.

Since the textbooks follow the English National Curriculum, the material, and the sequence in which it is covered, are similar to what any primary school is likely to teach in the relevant school year. Hence, these extension materials should be of much wider interest. Moreover, because the problems cover later Key Stage 2 content in

slightly unusual ways, the problems serve as an invaluable resource for Years 7 and 8 – revising and revisiting familiar material in a way that is both fresh and stretching.

The whole thrust of each set of problems is to avoid 'accelerating' to more advanced topics, and to cultivate a willingness to tackle unfamiliar problems in a way that forces students to think more deeply about standard material. This inevitably confronts the two F-words – frustration and failure – which are an integral part of the appeal of serious mathematics, but which are often (mis)construed negatively. Good problems often show that this can be just as unsettling for adults as for children. So it is important for teachers and parents who use this material to tackle the problems before using them, to experience for themselves how unsettling this encounter can be. And then to allow students the experience of suffering in the same way – experiencing something of the frustration and struggle (and eventual success) for themselves!

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# The PRAC Taxonomy for Formal Mathematical Assessments

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**Keywords: conceptual versus routine thinking; creativity in mathematics; algorithmic procedures; assessing problem solving**

## Introduction

The recent changes in England's education landscape have left many universities with less assessment data on their applicants than before, and less data means less informed admission decisions. In part response to this, we created the *Test of Mathematics for University Admissions* (TMUA) for school students to be able to take as part of their university application process. The process of designing the TMUA – that is, designing a test that is valid and reliable, and that measures potential to succeed on a demanding undergraduate mathematics degree – led us to the research summarised here.

Our fundamental question is how to create such a test. But answering this is not easy, especially as existing initial studies (Darlington 2013) suggest there is a contrast between the nature of school-level assessment and university-level assessment, as explained below. This suggests that merely producing a test similar in nature to an A-level test without a deep understanding of the contrast is not a useful approach. As a result, we set about digging more deeply into the structure of formal mathematics assessment questions and we found a need to have a taxonomy of mathematics questions, both as a guide for us as we select questions for the tests and for our writing team as they craft questions. There are, of course, already a variety of taxonomies that have been used for creating mathematics assessments and we looked at several of these. However, we found after some trialling that they had significant shortcomings for our purposes. What is more, whatever taxonomy we use needs to take into account that tests at school level and at undergraduate level are based on significantly different content areas: it must allow us to classify questions on these very different papers. We need a taxonomy that is consistent across this boundary. The rest of this article addresses the question of designing a taxonomy that offers these affordances. If we can subsequently also classify questions at different stages of school mathematics using the resulting taxonomy, that would be a further benefit.

This paper presents our new taxonomy, which we are calling the PRAC Taxonomy. Some prior work in this area is discussed in the next section, and our taxonomy is presented in outline in the subsequent section followed by an example of its use.

## Existing taxonomies

Darlington (2013) considered a variety of taxonomies that could be used for classifying mathematics questions. One of these, the MATH taxonomy (Smith et al., 1996), was designed specifically for classifying undergraduate mathematics examination questions, and this taxonomy seemed to be the most useful for our requirements. The purpose of this taxonomy is to support "[construction of] formal

examinations which assess a range of knowledge and skills…", and it achieves this by classifying questions into eight categories as summarised in Figure 1.

| Group A | Group B | Group C |
|---|---|---|
| Factual knowledge | Information transfer | Justifying and interpreting |
| Comprehension | Application in new situations | Implications, conjectures and comparisons |
| Routine use of procedures | | Evaluation |

Figure 1: Smith et al. (1996) MATH taxonomy

To summarise Smith's MATH taxonomy, we might say that Group A includes routine questions, Group B consists of application questions, requiring students to apply what they know in somewhat new contexts, while Group C covers mathematical logic and proof in various guises. Darlington applied this taxonomy to a variety of school-level and undergraduate examinations, and showed that there was a significant difference between the balance of questions in the three groups between A-level and undergraduate papers.

Our initial idea was to create the TMUA papers by having a test with questions fitting into these three groups in a ratio that reasonably matched current undergraduate papers. However, multiple issues arose when attempting to do so. The first was that some of the categories (especially using the detailed descriptions and examples in Smith et al. (1996)) seemed specific to undergraduate course content and did not translate easily to A-level content. The second was that we struggled to articulate clearly the meaning of, or distinction between, some of the categories, in particular those in Group B. The third problem was that we struggled to write questions appropriate for school students for several of the categories, despite our best efforts and our desire to test mathematical thinking. This led us to develop our new PRAC Taxonomy.

## The PRAC Taxonomy

Our taxonomy classifies a question by quantifying the relative proportion of routine or algorithmic cognitive effort required to answer it and the relative proportion of creative or conceptual cognitive effort required. (This is the origin of the name "PRAC": "Proportions of Routine And Creative".) Other taxonomies consider the distinction between these two, but in a different way from ours, for example the *Key stage 2 mathematics test framework* (Standards & Testing Agency, 2015, pp. 29–32).

In addition, whilst classifying questions using the PRAC Taxonomy, the notion of "context" is crucial. Education is a process, and as such what is a routine task for an A-level student may well be a creative challenge for a younger student. For example, in our specific situation of designing the TMUA we understand the context, in part, by setting out a test specification – a summary of the knowledge that is expected of candidates. When classifying the questions, we then assume the target cohort is drawn from a group of students who are applying to study mathematics at university, so are likely to be reasonably confident with the content on the specification, and in particular with the routine procedures encompassed within it.

A naïve way to use this taxonomy would be to look at a question and to make an estimate of the percentage of the effort required which is conceptual/creative.

However, this turned out to be highly unreliable. The following procedure works much better, and is our current model under testing.

1. Work through each question in detail, including noting the thinking steps required (even if they would not normally be written down). The solution approach should be one that a student in the target cohort would reasonably be expected to use. (If there is more than one reasonable approach, write out solutions using each of the approaches.)
2. Break the solution into steps, where each step is either an algorithmic/routine block or a conceptual/creative block, and label each step as one or the other.
3. For each step, decide how much effort is involved in it (as explained below).
4. Total the effort required for the algorithmic/routine steps and for the conceptual/creative steps. This then gives the proportion of the total effort used for the conceptual/creative effort.
5. If there is more than one reasonable approach, this calculation is performed for each approach. It is likely that most students will take the "easiest" route, so the lowest proportion is used as the classification for the question.

When assigning a measure of effort to each step, the measure used should be consistent within each question. While it would be interesting to aim for some level of consistency between questions, this is significantly harder to achieve and not necessary for the purposes of this taxonomy. We have found that assigning a measure of effort of either $1$ or $\frac{1}{2}$ to each step generally works well: it seems hard to make finer distinctions in a substantially more meaningful way.

## An example of the PRAC Taxonomy in use

The following question appeared in the TMUA (2017, Paper 1, Question 13). It was an example of a question for which the naïve approach of estimating a proportion of conceptual/creative effort gave wildly different values, whereas the systematic approach was much more consistent.

> In the expansion of $(a+bx)^5$ the coefficient of $x^4$ is 8 times the coefficient of $x^2$.
> Given that $a$ and $b$ are non-zero positive integers, what is the smallest possible value of $a+b$?

This table shows the steps in a solution and an estimate of the type and amount of effort required for each step.

| Solution | Effort required with brief explanation<br>(R = routine/algorithmic, C = conceptual/creative) |
|---|---|
| Expanding gives<br>$(a+bx)^5 = a^5 + 5a^4bx + 10a^3b^2x^2 + 10a^2b^3x^3 + 5ab^4x^4 + b^5x^5$<br>so $5a^4b = 8 \times 10a^3b^2$<br>Divide by $5ab^2$ to get: $a^2 = 16b^2$<br>so $a = 4b$ (as $a, b$ both positive)<br>giving $a+b = 5$ when $a = 4, b = 1$ | R $\frac{1}{2}$: standard from A-level papers<br>R $\frac{1}{2}$: identify coefficients in question<br>C $\frac{1}{2}$: strategy to find smallest possible value of $a+b$ (not a standard question)<br>+ R $\frac{1}{2}$: standard manipulation<br>**Total:** R 1.5, C 0.5, proportion C = $\frac{0.5}{2} = 25\%$ |

In our current test construction, we are starting to take the proportion of routine and creative content into account, aiming for a spread of proportions across the paper as well as a spread of difficulties.

## Discussion

The PRAC Taxonomy is proving useful in our attempt to address our research question. It is effectively content-free, by specifying the context as part of the classification process.

We are in the early stages of reliability and validity studies of the PRAC Taxonomy. One encouraging sign is that an early trial TMUA test showed itself to be "very beneficial" as an admission assessment tool in a predictive validity study comparing performance on the TMUA and first-year undergraduate examinations (see Saad, Carter, Rothenberg, and Israelson (1999, p. 3-10) for more information on predictive validity and the scope of "very beneficial"). This suggests that this taxonomy may have captured something of the nature of undergraduate examinations without being dependent on the precise mathematical content.

Among our next steps, we plan to trial some questions with a group of students and to interview them about their solution methods. This will permit insight about the extent to which students tackle these questions in ways similar to those we presume they might do, and thereby help us to refine our model and approach.

We also aim to apply this taxonomy to other mathematics examinations to assess its wider applicability. In particular, we wish to assess a variety of undergraduate mathematics examinations to ascertain the proportion of high-creativity questions that it is appropriate to have on an admissions test if they are to be similar in nature to those of undergraduate courses.

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# Algorithms Alcatraz: Are children prisoners of process?

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**Keywords: primary mathematics; multiplicative thinking**

## Introduction

Multiplicative thinking is acknowledged as a key component of mathematics that underpins much of the mathematics needed beyond middle primary or elementary years. Researchers have found that children who do not develop multiplicative thinking are unable to understand higher level concepts such as fractions, proportional reasoning and algebra (Clark and Kamii, 1996; Siemon, Breed, Dole, Izard, and Virgona, 2006). Hence, it is important to understand what constitutes multiplicative thinking and to identify its key elements that can then be developed in a connected way across all school years.

Siemon et al. (2006) have defined multiplicative thinking as a capacity to work ***flexibly*** and efficiently with an extended range of numbers (e.g. larger whole numbers, decimals, common fractions, ratio and percent), an ability to recognise and solve a ***range*** of problems involving multiplication or division including direct and indirect proportion and the means to communicate this effectively in a ***variety*** of ways (e.g. materials, words, diagrams, symbolic expressions, written algorithms).

We have taken the liberty of highlighting three words in the above definition as they are particularly important. In order to work 'flexibly' with a 'range' of problems and communicate in a 'variety' of ways, we assert that there must be explicit teaching of the many connections within the broad idea of multiplicative thinking. Indeed, Anghileri (2000: 2) suggested that learning and practising procedures is not appropriate and that we must "focus on the links that demonstrate the logical structure underlying numbers and number operations". Here we explore the extent to which upper primary school students use written algorithms as a preferred method of computation.

## Multiplicative Thinking and Procedural Teaching

We suggest that the teaching and learning of traditional algorithms for multiplication as a procedure is not necessarily supported by the explicit teaching of the mathematics that underpins the use of such algorithms. Discussion around the learning of procedures, or as Skemp (1976) puts it, instrumental understanding, at the expense of relational understanding, is not new. He describes instrumental understanding as "rules without reasons", generally devoid of understanding in a conceptual way, yet the learning of mathematics in this way is often equated with true understanding by its proponents (Skemp, 1976: 2). In seeking to understand why this is so, Skemp (1976: 11) refers to one of the reasons for the preponderance of instrumental teaching as "the backwash of examinations". It seems reasonable to describe this in broader terms to include national testing. More recently, Hartnett (2015) described a study into the use of methods other than traditional algorithms and noted that it was difficult to entice students to 'give up' the methods they had already learned. Hartnett (2015: 288) suggests, "One possible reason was that they were successful with the algorithms and predicted they would not be as successful with something that was new and different".

## Methodology

We report on a study that developed from a larger and on-going study into multiplicative thinking of children from 9 to 11 years of age. The original study has been conducted for over three years in Western Australian primary schools and has gathered data from over 1000 children in eight schools. Two data gathering instruments – a written Multiplicative Thinking Quiz (MTQ), and a semi-structured interview – have been developed and refined during that time and are used in this current study involving two primary school classes at a school in Plymouth, UK.

The quiz was administered to both classes on the same day under identical conditions. In the MTQ, students were asked a total of 18 questions and several questions specifically asked them to show how they would calculate an answer. We wanted to find out the extent to which students employed a learned procedure or 'rule' or used the written multiplication algorithm in answering the quiz questions.

All students were interviewed within four days of completing the MTQ. They were asked questions that specifically required them to show how they worked out a range of calculations. Probing questions were asked on the basis of their responses to some of the MTQ questions and here we focus specifically on responses to the interview questions.

## Data and Analysis

During the interview, students were asked several questions that were designed to ascertain the extent to which they used an algorithm and partitioning, and how they could explain their thinking. The use of the interview questions in this way was purposeful in that we wanted to find out the extent to which an algorithm was used, whether it was used 'as a convenience' or in situations where it might not have been the best choice of computation method.

Five general observations were able to be made from the interviews with considerable consistency shown across the cohort (n = 56). First, most students chose to use an algorithm to calculate answers for number facts derived from 6 × 17, even when they knew that the answer was 102. The samples from student Johnnie (figure 1) is typical of a number of students during the interview.

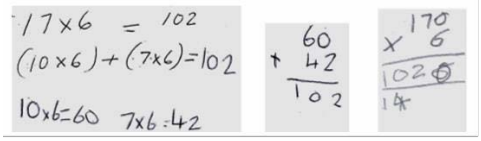



Figure 1: Samples from student Johnnie

Johnnie had already worked out the answer to 17 × 6 but persisted with using an algorithm to calculate the extended fact 170 × 6. He did the same when asked to work out the answer to 1700 × 6. Many of the students who were interviewed used the algorithm as first preference for calculating one-digit by two-digit multiplication examples. Johnnie needed to calculate the two parts of the partition and did not understand or trust the distributive property, even though the answer to 17 × 6 had already been established (from a previous question about the commutative property).

The second observation, as illustrated in Johnnie's sample, pertains to the partitioning and the distributive property. Once the distributive property was established, that 89 × 3 = (80 × 3) + (9 × 3), probing questions based on the distributive property were asked, such as, "What would give the same answer as (20 × 7) + (3 × 7)?" Many students were unable to make the connection and wanted to calculate the

two answers and add them together. The sample from student Brandon (figure 2) is indicative of this.

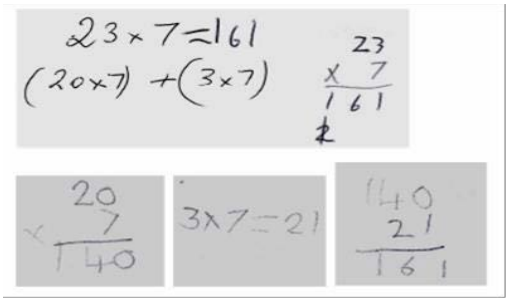



Figure 2: Samples from student Brandon

Brandon used the vertical algorithm to obtain the correct answer but said that he didn't know if the partitioned example would give the same answer so he worked out each part of the partition, again with a vertical algorithm for 20 × 7.

Third, most students used a vertical algorithm to calculate 23 × 400, with many of them writing the larger number on top. Many did that even after calculating 23 × 4 = 92. They persisted in using the algorithm when variations such as 2300 × 4 and 2.3 × 4 were given. Samples from Student Calvin (figure 3) are indicative of this.

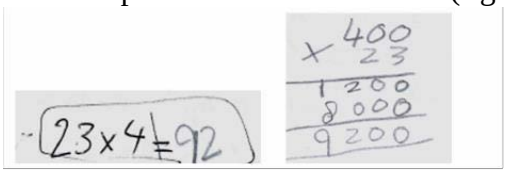



Figure 3: Samples from Student Calvin

Fourth, almost all students used the standard two-line vertical algorithm to calculate the answer for 29 × 37, although one used a four-line algorithm. Many also placed the 37 on top as it was the larger number.

The fifth general observation that can be made pertains to the use of a division algorithm for 200 ÷ 13. Several students referred to this as the 'bus stop method'. All but one of the students who were asked to work out the answer for this question began by writing the algorithm and then immediately wrote the multiples of 13 at the side. No student in the sample attempted to work out the answer in any way other than using the algorithm. A sample from student Ian is shown in figure 4. To follow up the 200 ÷ 13 example, he was questioned as to how he would do 200 ÷ 3. It is interesting that he used the same procedure when it would be reasonable to expect that he would not need an algorithm but could work it out mentally.

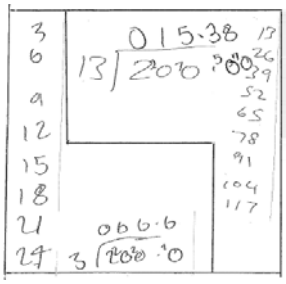



Figure 4: Sample from Student Ian

Student Robbie demonstrated understanding about how digits moved one place each time a number is multiplied or divided by a power of ten. However, in the following sample and interview he opted to use algorithms and talked about 'the number of zeros behind the decimal point' instead of employing his understanding of digit movement.

R: [24 × 0.06] I would divide it [the 24] by 100 so it would be 0.24 by 6.

I: Why would you divide it by 100?

R: Because the equivalent of that [0.06] is 6 divided by 100. [He set out a vertical algorithm]. You've got to remember that there are two numbers behind the decimal point so it is 1.44.

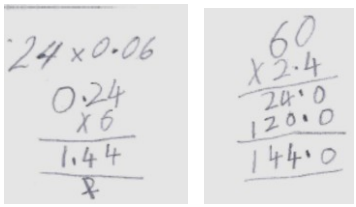



Figure 6: Samples from Student Robbie

## Conclusions

One point we want to make here is that we have no objection per se to the teaching and use of algorithms such as those described. Our concern is that many students opt to use algorithms when there is no need, and often at the expense of mental computation strategies. The evidence from this relatively small sample suggests that the flexibility of Siemons et al. (2006) may be compromised by a preference for using algorithms. We believe that it is important for students to understand the mathematics that underpin algorithms, 'rules', and procedures and that mathematics should be explicitly taught. Also, it is important for algorithms to be used as one way of calculating along with a wide range of mental and other strategies, so students can avoid becoming 'prisoners of process'.

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# Finding the adult in adults learning mathematics: an academic and political study

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**Keywords: numeracy; adult education; non-traditional student; lifelong learning; ICME13**

## Adults learning mathematics

This paper sets out to establish the importance of recognising the adult and further education sector as a significant contributor to mathematics education research. It builds on the work done at the International Congress on Mathematical Education (ICME) over a number of years. This particular study arises from ICME13, held in 2016.

At the ninth British Congress of Mathematics Education (BCME 9) a number of researchers belonging to ALM presented a range of papers in order to draw greater attention to this sector. It is important to note that within mathematics education research the adult sector is a very tiny minority. Alongside this, those of us working on adult mathematics are also aware that within the sector of adult and further education, mathematics is a very small minority.

Adults Learning Mathematics (ALM) is an international research forum bringing together researchers and practitioners in adult mathematics/education in order to promote the learning of mathematics by adults. It has now been active for 25 years and continues to promote research and develop practice through various activities that can be found on the website http://www.alm-online.net.

## Adult Numeracy

I have for some time investigated what 'numeracy' is (Kaye, 2003).There are many definitions and uses of numeracy but what they have in common includes: context, relevance to real situations, used for solving problems, personal choice of methods and favours personal empowerment. Mathematics, or mathematical, is always mentioned to identify the nature of the activity, but something else is always there as well. As Dave Tout (1997: 13) said, "we can say that numeracy is not less than mathematics but more".

In looking at meanings of numeracy it becomes obvious very quickly that although the concept is very familiar, there are many terms used to describe something very similar. The terms found most often are: quantitative literacy, mathematical literacy (figure 1), school mathematics, functional mathematics and maths.

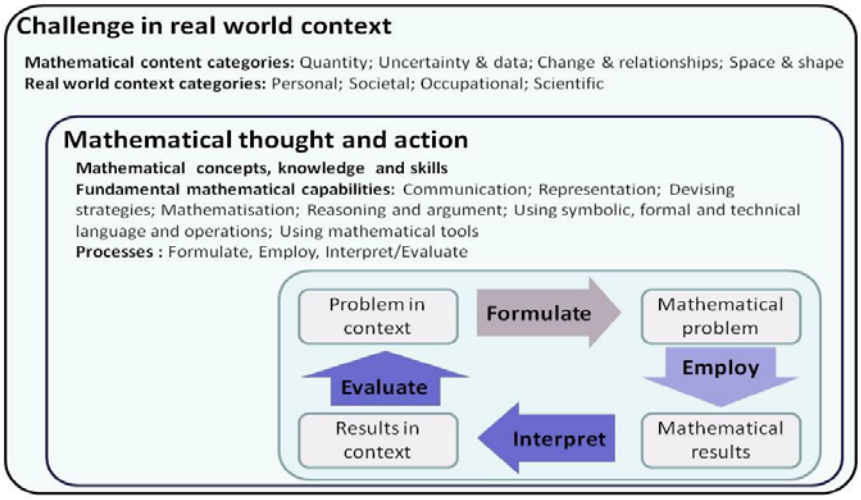



Figure 1. A model of mathematical literacy in practice for PISA 2012 (Tout and Gal, 2015: 5)

## Lifelong Learning at ICME13

At ICME13 special interests were represented by over fifty Topic Study Groups (TSGs) and the adult sector was represented in a few of them, but most specifically by TSG 6 *Adults learning of mathematics – lifelong learning* that was described on the ICME website as:

> The Study Group encompasses all mathematics and numeracy education undertaken by adults for the purposes of personal, social, political or economic development, and as a course of study in its own right, or in support of learning another subject, developing a skill or furthering an activity.

The TSG on lifelong learning published an associated preconference Topical Survey (Safford-Ramus et al., 2016) that set out many of the main debates associated with adult mathematics education research.

Attendance at TSG was my main purpose for participating in ICME13, as the content of many of the other presentations took my interest and led me to change my focus and develop an alternative research question as I interacted with the conference with my 'adults learning mathematics' hat on.

## School, university and . . . ?

Over the course of ICME13, I found the issue of teaching 'adults' mathematics a far more significant issue to consider than my on-going investigations into the use of the term 'numeracy'. I began to think more about **who** is being taught, and in what institutions or contexts, than how to describe **what** is being taught. Within school and university contexts the question of what is being learnt can be answered by reference to curricula, syllabi and degree regulations. However, this is not necessarily the case when considering adult learners. In reflecting on this situation I have realised that the need to define the scope, meaning and significance of the content of **adult numeracy** requires consideration of 'adult' as much as 'numeracy'.

## Where is the adult?

My focus shifted to looking at mathematics education research of direct relevance to adults, but where adult learners were not acknowledged at all. As this investigation

progressed I became aware that other researchers had considered this in other contexts and the term 'non-traditional student' was sometimes used where I might use 'adult'.

I reviewed the whole of the ICME13 scientific programme, using both the website and the printed programme distributed at the conference. I also made use of the Topical Surveys associated with some of the Topical Study Groups. I give below two of the most significant examples:

TSG 28 entitled *Affect, beliefs and identity in mathematics education* addresses broad human experiences that connect to the concerns of those who focus on adults' education. However, looking in more detail at the Topical Survey associated with TSG 28 (Golden et al., 2016) there is some suggestion that the focus is still associated with a traditional school or college student. This is so disappointing, it gave rise to my conclusion that this is a case of the 'hidden adult'. The relevance of affect and motivation are so significant to the adult learner. What is all the more striking about this commentary is that in the introduction, which lists topics that are not addressed in this slim volume is 'mathematics anxiety'. This is a topic researched widely in adult mathematics education research (Evans, 2000: Chapter 4; Klinger, 2008; Safford-Ramus et al., 2016). Yet the two references given are to neuroscience, one of which does not even refer to mathematics. In my view, this lack of regard for adult learners diminishes the scope and validity of the research.

My second example is from TSG 34 *Social and political dimensions of mathematics education – current thinking* and its Topical Survey (Jurdak et al., 2016). The topic is a social and political analysis of mathematics education that mainly references the school sector. Yet to me such an analysis is at the heart of adult mathematics education research. For example it states:

> Mathematics education is a social institution which is inseparably linked to power. Mathematicians and scientists, education researchers, politicians, teachers, students and parents are interested in mathematics education for various reasons, for example for the recruitment of future specialists, for the education of the enlightened citizen, for the vitality of the state economy, for the pursuit of a meaningful and dignified purpose in life or for the allocation of beneficial opportunities in further education and work. (Jurdak et al., 2016: 10)

This could not be a clearer statement about the research and practice of adults learning mathematics. Yet the context in which the statement is presented makes no reference to this sector of education, nor acknowledges the existence of a body of knowledge built on the research of adult mathematics education. Not only is adult learning missing from the scope of the research but the analysis would be greatly enhanced by the inclusion of the adult or non-traditional student experience. By remaining in the bunker of school education the conceptual growth of the topic is limited. This is yet another example of the 'hidden' or maybe even the 'invisible adult'.

## Conclusion

My conceptualisation of the research problem is still in formation. I have used an overview of some research programmes about learning mathematics that put a strong emphasis on the social, political, cultural and personal aspects of the mathematics learners' lives from ICME13 (2016). I have always considered these aspects of education research to be a core part of the research into adults learning mathematics (or lifelong learning, or adult mathematics education). Therefore it is disturbing to find that such research does not only fail to recognise the body of work developed under the heading of 'adult', but makes claims that exclude the adult learner. I believe the overt

recognition of the adult learner and the associated body of mathematics education research knowledge is essential.

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## Using NRICH tasks to develop resilient problem solvers

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**Keywords: Resilience; Problem-solving; Rich Tasks**

With the new national curriculum, GCSE and A levels, and the demand from universities and employers for resourceful and independent problem solvers, developing our students' problem solving skills is more important than ever. In our workshop for secondary teachers, we shared some new NRICH resources (nrich.maths.org) which focus on developing resilience.

We have been thinking a lot about resilience since attending a symposium led by Clare Lee and Sue Johnston-Wilder. This led to a presentation at the 2017 Mathematical Resilience Conference, where we began to explore ways of scaffolding NRICH resources. Our *Being Resilient* collection emerged from the discussions that followed. Clare and Sue have since contributed an article to our Spring 2018 NRICH Secondary feature *Resilience and the Growth Zone.*

We began our workshop by inviting delegates to have a go at questions on worksheets from two of NRICH's Short Problem Collections. We then collected together everyone's different strategies from when they got stuck, and made an *I'm Stuck!* Poster (figure 1).




Figure 1: Poster showing strategies suggested by delegates for getting unstuck.

Next, we introduced three problems that first appeared on NRICH some years ago, which have recently been revised. We explored the three problems without scaffolding first, and then looked at the new versions for each. The three original problems can be found at https://nrich.maths.org/1821. For the first problem, students are invited to answer the questions, "What is the smallest number with exactly twelve divisors?" and "What is the smallest number with exactly fourteen divisors?" In our new version, *Counting Factors* (https://nrich.maths.org/480), the problem starts by examining different ways to count the factors of a number, to encourage students to look at the structure of numbers and their factors, before asking the specific questions

about finding numbers with a particular number of factors. We also chose to change the language from 'divisors' to the more familiar 'factors'.

In *Iff*, the original problem posed the question, "Prove that if $n$ is a triangular number then $8n+1$ is a square number. Prove, conversely, that if $8n+1$ is a square number then $n$ is a triangular number." The revised problem starts by inviting students to explore what happens when they multiply a triangular number by 8 and add 1. It goes on to offer two Proof-Sorters, one for the first statement and another for the converse. These are available as interactivities as part of the Teachers' Resources on the website: https://nrich.maths.org/790/note.

The final problem we shared was *Areas and Ratios*. The diagram from the problem is shown below:

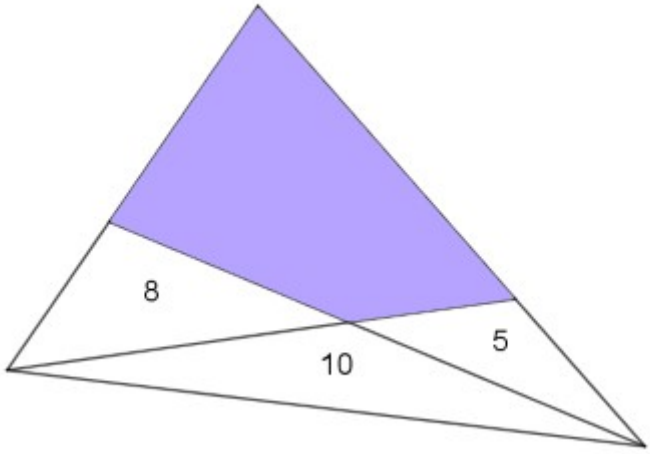



Figure 2: Areas and Ratios – Find the shaded area.

The problem originally consisted of the diagram and an instruction to find the shaded area. Our scaffolded version of the task contains a sequence of suggestions, hidden behind 'hide/reveal' buttons on the website, which act as hints towards fruitful lines of enquiry. We discussed the possibility of printing these hints out on cards to give to students if they are stuck.

Once delegates had worked on the three tasks, we discussed the following questions:

- What did it feel like to do a problem with the scaffolding?
- Did you feel a sense of achievement?
- Did the scaffolding take the fun away?
- Low-achieving students deserve the chance to work mathematically - being a mathematician should be an experience everyone can have. What might we need to do differently to enable them to do so?

More tasks with suggestions for similar scaffolding, together with other useful links, can be found on the NRICH website at nrich.maths.org/bcme2018sec.

# Preparing Students for Mathematical Olympiads

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**Keywords: Mathematical Olympiads; British Mathematical Olympiads**

## Olympiad Competitions

The United Kingdom Mathematics Trust (UKMT) runs several competitions for individual schoolchildren. The age groups are Junior (Years 7 and 8), Intermediate (Years 9 to 11) and Senior (Years 12 and 13). The initial rounds are multiple choice Challenges that are machine-marked. The most successful candidates are invited to take the relevant Olympiad paper. The next most successful candidates are invited to take the relevant Kangaroo paper, which are multiple choice and machine-marked.

Years 12 and 13 take the British Mathematical Olympiad (BMO) papers. Round 1 consists of six questions and Round 2 consists of four questions. Each paper is $3\frac{1}{2}$ hours. They are used to help select a team of six to represent the United Kingdom at the International Mathematical Olympiad (IMO), which takes place each summer.

## Knowledge Required

Full written solutions are required, not just answers. Complete proofs are needed for any assertions made. But candidates may quote without proof any standard results, such as Pythagoras' Theorem, which are well known to schoolchildren. A complete solution to one question will gain more credit than unfinished attempts at several questions. The earlier questions tend to be significantly easier.

Despite being aimed at sixth-formers, no questions need Calculus, Complex Numbers, advanced Trigonometry or Probability. Most of the questions can be done using only GCSE knowledge, albeit used at a deep level. The following areas are perhaps exceptions.

### *General Ideas*

Proof by induction is sometimes required, but not a formal A level style write-up. The candidate needs to explain how a problem can be reduced to a similar problem of a smaller size.

Many problems naturally fall into two parts. The most obvious case is an "if and only if" proof. The candidate must clearly start from the left-hand statement and deduce the right-hand statement. Then they must start from the right-hand statement and deduce the left-hand statement.

Solving equations is a less obvious example. It is not enough to deduce a list of possible solutions. The candidate must show that all these solutions actually work in the original equations. Obvious methods of solving equations often lead to false extra solutions.

Similarly, to find an upper or lower bound, it is not enough to show that an inequality holds. The candidate must also construct an explicit example to show that the inequality cannot be improved.

Questions are never meant to contain irrelevant information (apart from obvious window-dressing such as names). If a candidate has not used all the information given in the question, it is likely that their attempted solution is bogus.

***Algebra and Inequalities***

Some algebra questions are disguised as arithmetic. For example, we may readily evaluate $2019^2 - 2 \times 2018^2 + 2017^2$ by introducing $n = 2018$ and expanding brackets. Careful use of symmetry often simplifies the algebra.

It is worth knowing the formulas for Arithmetic and Geometric Progressions. For BMO2 it is worth knowing the Arithmetic-Geometric mean inequality, namely $\frac{1}{n}(a_1 + a_2 + \cdots + a_n) \geq \sqrt[n]{a_1 a_2 \ldots a_n}$ when the values are all positive. For BMO1 is should suffice to know the Triangle Inequality (each side of a triangle is shorter than the sum of the other two sides) and the Discriminant Inequality (if a quadratic has real solutions then $b^2 \geq 4ac$).

***Combinatorics***

Combinatorics is the art of counting possibilities, which is often applied in probability. It is not just pattern-spotting, but the systematic enumeration of some small examples often leads to a conjecture or a recurrence relation, which can then be justified rigorously.

It is worth knowing about Binomial Coefficients. The Pigeonhole Principle often appears (if there are more letters than pigeonholes, then at least two letters share a pigeonhole), as do Colouring Arguments (for example, prove that a chessboard cannot be covered by 31 dominoes if two opposite corners are removed).

***Geometry***

This is an area where UK students are weak compared to overseas students. Gerry Leversha's book *Crossing the Bridge* is an excellent introduction at a suitable level. Many BMO1 questions can be done using only GCSE knowledge (Circle Theorems, Similar Triangles and Triangle Areas). But these will be applied in general situations, rather than calculating a particular case. Triangle Congruence Conditions are often required, but they are not always taught at GCSE.

Diagrams are never given; the first step should be to draw one. It should be accurate enough to reflect all the information given in the question, but not so accurate that the candidate assumes lots of things to be true which are **not** given in the question. For example, if the question requires proving that three points lie on the same straight line, it is better to draw the diagram as if they are **not** collinear.

***Number Theory***

Number Theory is about dealing with quantities that are known to be integers. For BMO1 the key ideas needed are Parity (even and odd numbers), Binary and other Base-Arithmetic, and Integer Factorisation. For example, to find integer solutions of $2x^2 - 3x - 4 = 0$ (a Diophantine Equation), first rewrite the equation as $(x - 3)(x + 2) = 10$ then consider all factorisations of 10 (including negative integers). For BMO2, a familiarity with Modular Arithmetic is sometimes expected (although this is equivalent to working with the units digit in base $n$).

## Olympiad Mark Schemes

Olympiad mark schemes are very different from most examination mark schemes. Each question has two separate mark schemes. The 10- mark scheme is applied to solutions that are nearly complete. Penalties are applied for arithmetic errors or other minor errors, but usually the lowest possible mark is 7 out of 10. The 0+ mark scheme is applied to solutions that are missing at least one crucial ingredient. Bonuses are awarded for insights, which lead towards a solution known to the markers, but usually the highest possible mark is 3 out of 10. It follows that marks of 4, 5 or 6 are usually impossible. This is why one complete solution may gain far more than several partial solutions.

The most important part of the mark scheme is the 10- criterion that determines which mark scheme to apply. For example, in solving a Diophantine Equation, a typical 10- criterion might be as follows: "The candidate successfully reduces the problem to a finite list of pairs (□, □) to check. Condone minor arithmetic errors but not algebraic errors. Condone failure to consider negative factors."

## Resources

The BMO website (https://bmos.ukmt.org.uk/) contains over 40 years of past BMO papers, with video solutions of recent BMO1 papers. The UKMT website (https://www.ukmt.org.uk/) contains the most recent Mathematical Challenge papers and other resources. In particular the UKMT publishes a series of books containing past Olympiad papers for the various age groups, together with hints, solutions and extensions.

Ultimately, the only way to learn how to solve hard mathematics problems is to try to solve lots of hard mathematics problems. Reading through someone else's solution gives little benefit until one has spent a serious length of time grappling with the problem on one's own.

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# Learning mathematics with origami

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**Keywords: origami, practical geometry, problem-solving**

## Introduction

This paper presents a selection of activities, some from Lam and Pope (2016). Most of the models used are purely geometric and do not represent objects. However, we start with a representational model that illustrates how any origami model can be an opportunity for using and applying mathematics.

## Randlett's *Flapping Bird*

The traditional *Flapping Bird* is well-known (Harbin, 2003) but has some disadvantages: acceptable results need accurate folding and includes a "petal fold" which beginners can find challenging. The *Flapping Bird* by Randlett (1963) is more forgiving of inaccurate or soft folds. The flapping action works well and will not rip no matter how vigorously flapped. The only step that is hard to describe is the squash fold when making the wing and neck, so we teach this model by direct instruction.

Any model can be unfolded back to the starting paper. The set of lines and their direction (concave valley folds and convex mountain folds) may appear complex but obeys some geometrical rules (Hull, 2002). However, for primary and secondary school mathematics, some accessible tasks that require reasoning:

- identify the angles between lines, and determine their magnitude
- find specific shapes e.g. trapezium or isosceles triangle

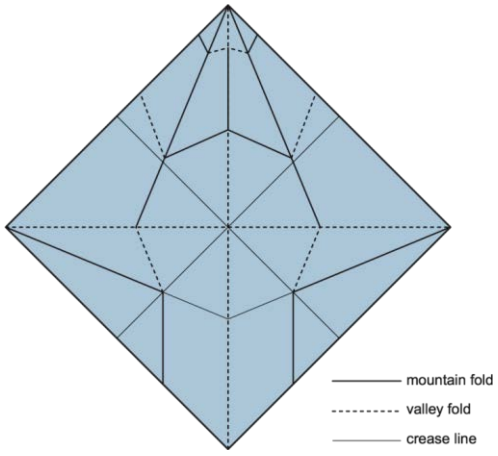



Figure 1. Crease pattern of Randlett's Flapping Bird

## Reverse Engineering

For a contrast of learning method and origami style, we ask participants to work out how to make a model themselves (Robert Neale's *Pinwheel-Ring-Pinwheel*, Figure 2). We give each small group two models: one to be left intact and the other to be carefully

taken apart and analysed. Modular origami is well suited for this approach as participants will find better methods and improve accuracy as they make more units.

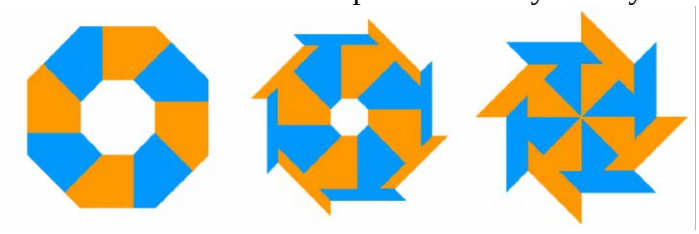


Figure 2. Neale's *Pinwheel-Ring-Pinwheel* (also know as *Magic Star)*

Figure 3 shows a traditional model made from two rectangles. The units are mirror images which can surprise and puzzle participants trying to make their own. Year 7 pupils made this poster to communicate how to make the star to a different class. This gives a genuine purpose to the work, and they find out how effective their poster is when the other class either succeeds in making the star, or fails to do so.




Figure 3. Instructions by year 7 pupils for folding the *Shuriken*.

Figure 4 shows a classic 3D model that uses the same starting point as Randlett's *Flapping Bird*.

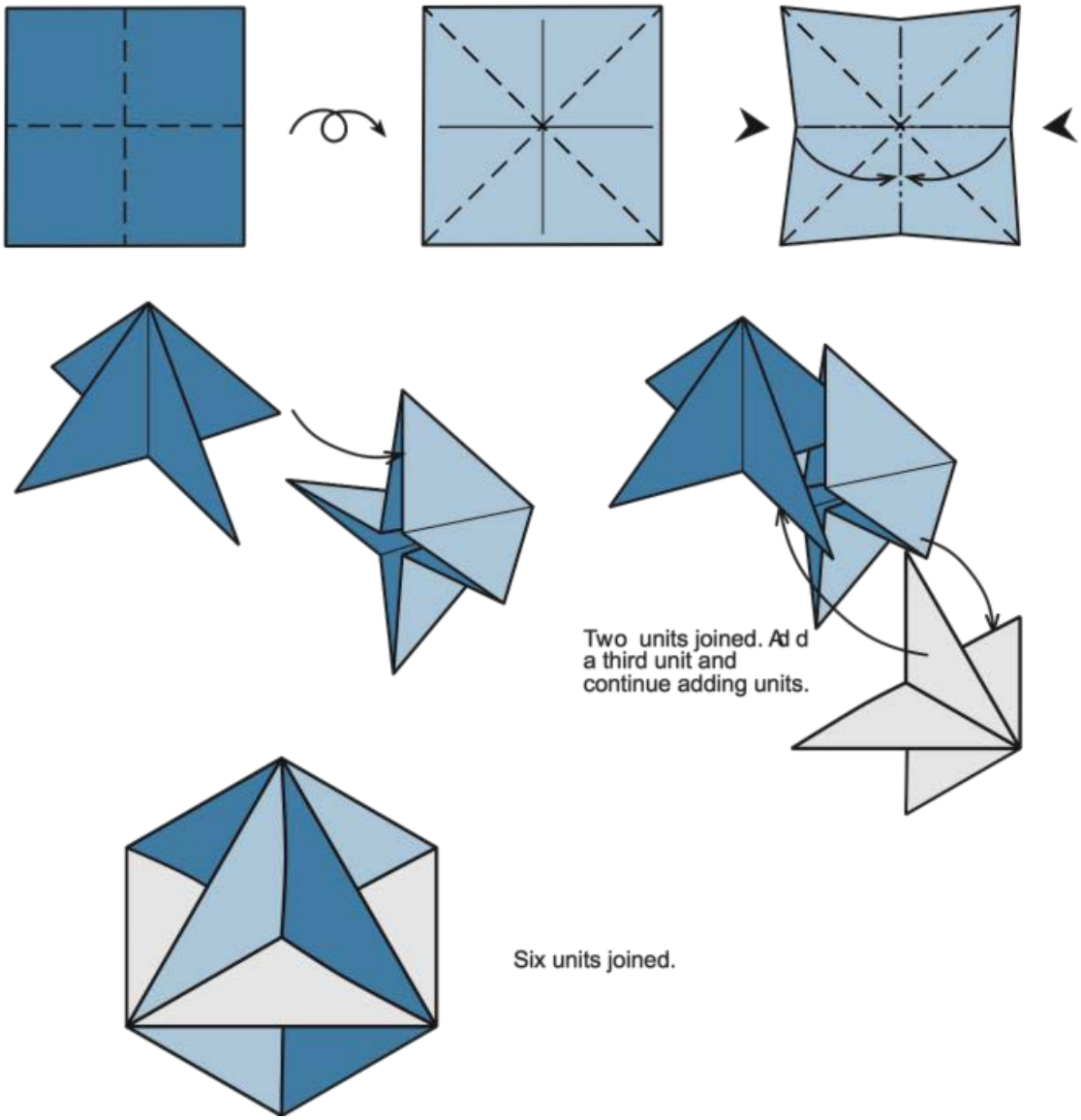



Figure 4. Instructions for making Robert Neale's *Skeletal Octahedron* from six squares of paper.

Figure 5 shows how to assemble Paul Jackson's *Cube*. The units are made by folding the sides a square to the middle, and then folding the shorter side to the middle.

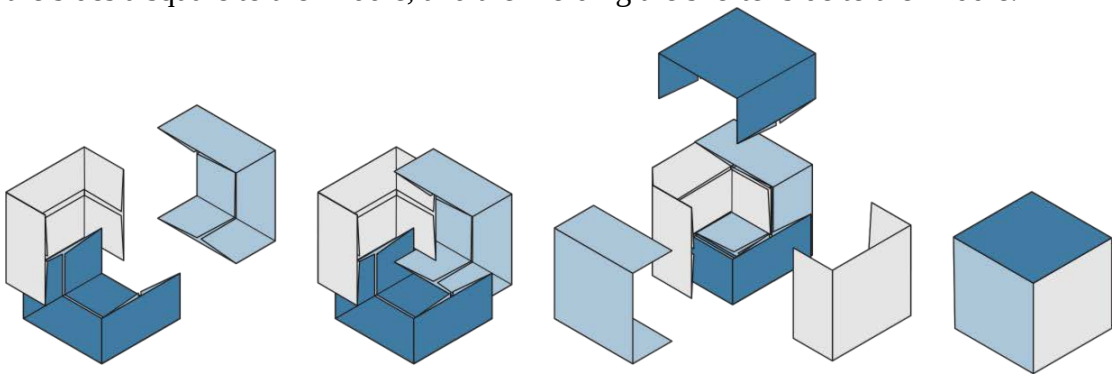


Figure 5. Assembly of Paul Jackson's *Cube*. Left two steps: joining three units. Right two steps: adding the last three units to make the cube.

## Hexagram

It is impossible to trisect an angle using only an unmarked straightedge and compass. However, it is possible to trisect an angle of 90° or less by folding. Figure 6 shows the special case of trisecting 90°. Bisecting the 120° angle can be a valuable learning experience as it focuses attention on the salient features of angle bisection, the two sides and the point at which they meet. A common mistake is to make the fold line pass through the upper right corner, even though that fold will not bisect the angle.

Making the *hexagram* requires finding the centre of the equilateral triangle. Depending on the learning objective, this can be found using minimal creases (using the flap made in step 1) or by folding the three lines of symmetry of the equilateral triangle. Folding the three corners to the centre makes a regular hexagon, which is $\frac{2}{3}$ of the original triangle. With a few more folds the hexagram is possible.

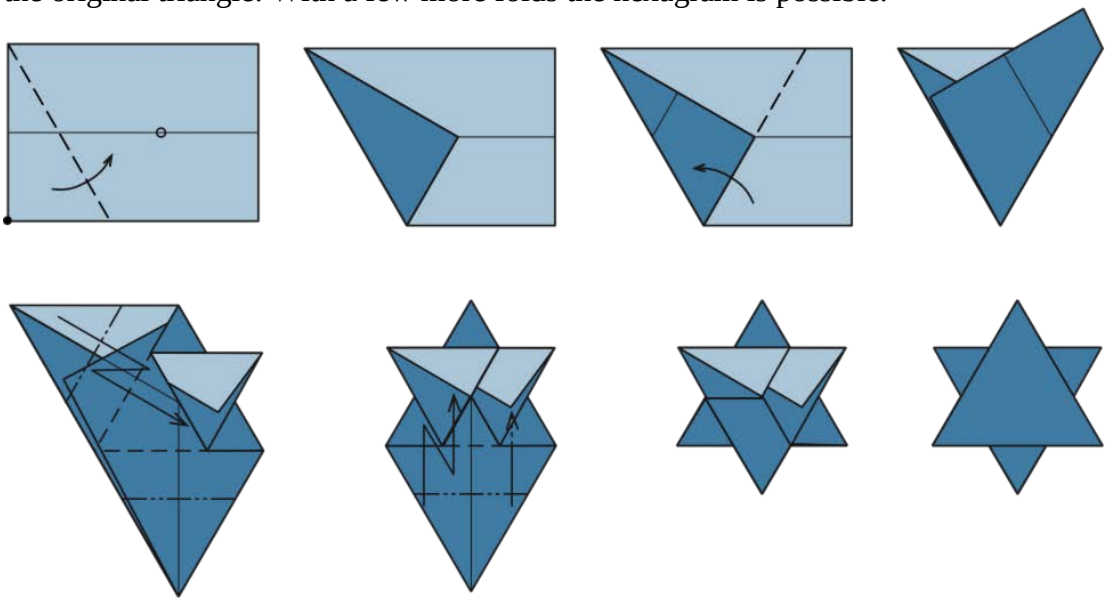


Figure 6. First and last steps for making a hexagram. Top row: first steps for folding an equilateral triangle from a sheet of A4 paper. Bottom row: last steps for folding a hexagram.

## Conclusions

This paper has presented some activities for specific origami models. However, the approaches described could be used for almost any origami model.

Keen folders may wish to vary, extend and generalise models (Lam, 2018).

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# Mathematical Resilience Workshop

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**Key words: Mathematical Resilience; emotions in learning; perezhivanie; perseverance**

## What do we know about Mathematical Resilience?

Mathematical Resilience is a positive stance towards learning mathematics, which is characterised by:

- having a growth mindset – the resilient learner understands that they can improve their own mathematical understanding through their own efforts and appropriate support from others. They do not accept that they cannot do mathematics, although they will readily agree that they cannot do certain areas of mathematics yet!
- understanding the personal value of mathematics and that they are a valued member of the community of mathematics learners. This is ultimately about inclusivity and stands against the elitist idea that only some people are able to do mathematics and if you cannot it does not matter. Resilient learners understand that mathematics will empower them; being willing to engage with numbers prevents many ways that unscrupulous people try to defraud or cheat; the systematic thinking, the logic and reasoning required allows them to problem-solve in their lives and careers. Resilient learners know mathematics has value and they also know that their ideas and understanding are valued and useful as the community of which they are part struggles to learn mathematics.
- knowing that mathematics requires struggle to learn and that with the right support success will come. The idea that mathematics, with its abstractions and symbolism, can throw up barriers is understood, and a resilient learner will accept challenge and persevere until they succeed, as they have succeeded before. Resilient learners do not expect their teacher or peers to smooth the path, rather to support them as they overcome difficulties.
- understanding how to seek and get appropriate support that will enable learners to succeed. This may mean talking to adults and peers in their groups, and therefore they can expect their teacher to help them to develop language suitable for talking about mathematical ideas. However, it also may mean accessing help from books or from the internet. Someone mathematically resilient knows they have the right to understand. Hence, if one form of support does not work they seek another.

Working to develop mathematical resilience is important as a result of myths that have circulated within UK society and in many other countries. For example, when faced with mathematics, many people believe they have to remember how to do each separate move in a process and to reproduce these moves substituting different numbers. They feel they just cannot remember all the moves, therefore they 'cannot do mathematics'. Others feel that mathematics is something that only an elite can do and they are not part of that elite so once again they do not expect to be able to do mathematics. A further common myth is that only the teacher knows the one 'right

way', and therefore asking other people will not help if they cannot understand what the teacher says.

However, we think that mathematics is a connected subject. Therefore if you can do one bit you can make a start on doing something else. We consider that asking someone to 'remember' and especially to 'remember quickly' is counterproductive as it often provokes a 'freeze, fight or flee' response, whereas giving time to think and calmly remember or reconstruct is likely to result in success. We also consider that everyone is able to improve their understanding of mathematics, with effort, help and support. Everyone can become sufficiently numerate, unless they have a rare learning disability, to prevent them being easily defrauded and in that way mathematics can be useful and empowering for everyone.

We argue against 'one right way', we contend that there are many, many ways to approach any mathematics problem. We understand that some ways are more efficient and elegant than others but if the learner uses their way to get the answer then it is good enough for now. Of course we want learners to develop efficient and fluent ways to engage with mathematics but we believe that such ways are the result of being willing to seek understanding and apply reasoning. Learners need to feel part of a community where it is safe to make mistakes and to acknowledge that they are stuck. Such a community offers just the right amount of help and support so that the learner knows that they succeeded. Resilient learners are able to experience the joy of overcoming barriers and succeeding.

### *Group 1: Building Mathematical Resilience using awareness of perezhivanie*

Participants worked with us on what Vygotsky had to say about emotion in relation to cognition, environment and learning effectively, to reflect on experiences of learning and teaching mathematics and develop our ideas about mathematical resilience. We introduced Vygotsky's notion of 'perezhivanie' as involving a significant experience, which determines the influence of the physical and social environment in later events and which can be a powerful tool for reintegrating emotion and cognition.

> An everyday reading of perezhivanie brings to our attention what is happening in the environment, and how the situational characteristics are being experienced by the [learner] (Fleer 2016: 36).

Perezhivanie is an integrated unit: on the one hand, it involves the environment, that which is being experienced, on the other hand, how that environment is perceived (Vygotsky, 1978). We asked participants to reflect on how much account is taken of perezhivanie in mathematical learning by both teachers and learners. We used the example of Bonnie, who had a preference for drama and her personality was outgoing and empathetic; she was a confident mathematical learner until she was 14. That year she found her mathematics classroom inhumane. The teacher seemed to lack empathy, and Bonnie has been frightened of mathematics since then. Her previous perezhivanie was relatively positive which indicates her experience of that mathematics environment was very significant and resulted in a changed perezhivanie in relation to mathematics.

Developing mathematical resilience may involve a complete transformation of the learner's perezhivanie by changing the prism through which the learner sees and understands her situation. This is much more than just adopting a positive approach as there is extensive emotional struggle involved.

> The concept of perezhivanie is intriguing for educators as it raises questions about the totality of the lived experiences of students within our classrooms and how

> these experiences might be cultivated for the most meaningful forms of learning.
> (Davis and Dolan, 2016: 51)

In the discussion, we noted that the concept of perezhivanie and the mutual relationship between cognition and emotion was seen as fairly new. Some delegates expressed their own experience of perezhivanie as students when they had found some mathematics teachers cold and without empathy. Some felt that they have grown into cold mathematics teachers as a result. For example, one teacher reported finding it extremely difficult to adapt his approach. He had been told he is cold and clinical by one of his students, who reported feeling as though they are being taught by a 'patronising teacher'. He does not want to be like this. He realised that this student had regressed in mathematics and he provided them with one-to-one support to help gain back their interest. This raised the question how to support teachers to transform their own perezhivanie.

During the group work, similar conversations were observed about the participants' mathematics departments' policies, linking them to the idea of perezhivanie. An example was given of students receiving lunch time detentions due to lack of homework when the lack was because students could not access the work. Conversations moved towards teachers' intervention and support to transform students' perezhivanie. The question was posed, "To what extent can the teacher provide intervention and support to transform the situation?" Support should enable learners to reach the stage at which they can overcome both their cognitive and emotional difficulties and, therefore, regulate their own learning and emotions. Points which may be worth exploring further are: the collaboration between teacher and learners to transform the perezhivanie (not only the teacher giving direction and the learners accepting and following them), how teachers can support learners to develop skills to transform their perezhivanie independently and learners' past perezhivanie due to summative assessments and how can teachers support them to transform this.

### *Group 2: Perseverance: balancing challenge and support*

This workshop explored two issues related to developing mathematical resilience: the need for learners to understand how to work at mathematics – particularly a willingness to struggle and persevere (Bandura, 1994) and the need to carefully balance challenge with support and scaffolding to avoid anxiety and disengagement.

The need to balance challenge and support can be represented through the Growth Zone Model (Lee and Johnston-Wilder, 2013). For many learners, studying mathematics is a source of anxiety – which can lead teachers to over-support due to a keenly felt wish to reduce the discomfort of learners. Such over-support tends to keep learners within their comfort zone and reduces the potential for new learning. In contrast, not supporting enough means learners are allowed to flounder, which is likely to take them into their red zone – anxious, frustrated and disengaged.

The solution lies in the teacher stepping back, being less directive and letting learners struggle and take responsibility for their own learning, while accessing the support of peers and others. It also requires teachers to be sensitive to the needs of each learner, judging when scaffolding is required, and to create a positive classroom climate where all learners feel emotionally safe and are able to ask honestly for what they need to thrive.

Motivation to persevere is also part of Bandura's (1994) notion of self-efficacy. The more students feel self-efficacious the more they will persevere. Bandura's four sources of efficacy beliefs are: mastery experiences – the experience of success, and

particularly success achieved through effort and struggle; social modelling – observing role-models who have achieved mastery despite adversity; social persuasion – having others who provide opportunities for mastery in a safe and purposeful manner; and physiological states – how we interpret our emotions in response to stress, struggle, or mistakes. Workshop participants were asked to reflect on these beliefs whilst discussing how their actions might influence learners' self-efficacy a) when introducing a task, b) during the task, c) while reviewing the task. Suggestions included:

| **Before** | **During** | **After** |
|---|---|---|
| Establish a positive/'safe' atmosphere where learners feel happy to make mistakes, answer questions and get stuck<br>Set and maintain ground rules for behaviour and ways of working<br>Provide clear criteria for success (which should be process focused)<br>Ensure clarity – so learners can explain what they need to do<br>Ensure teachers know which pre-requisite knowledge and skills are required to enable learners to succeed and have ways to establish that the students have that knowledge or those skills<br>Provide low threshold-high ceiling tasks that are accessible to all | Allow quiet thinking time at the start<br>Use exploratory questions (Have you seen a similar problem before? Are there any resources that would help you? What strategies have you tried or could you try?)<br>Ask learners to talk through and share approaches that they have tried<br>Provide prompts around the room to refer to (e.g. Brain, Book, Buddy, Boss, or C3B4ME – see three before me)<br>Provide a 'help desk' or 'tool box' of resources for learners to draw on<br>Praise processes and effort (not results or answers)<br>Ask learners to convince each other that their methods or answers are appropriate<br>Encourage flexible thinking to avoid getting 'in a rut' | Time for reflection – make this important!<br>Reflect and unpick the processes used – identify what worked well (What did I overcome? How did I get unstuck?)<br>Share ideas and normalise error-making as part of the process<br>Encourage learners to make videos of how to solve a problem/overcome a difficulty<br>Make links to similar problems<br>Feedback on the process – not just the solution<br>Recognise success – how is it marked or identified so you can look back at it?<br>Plan ahead - what will I do next time?<br>Celebrate success! |

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# Progression towards open access environments in the teaching of statistics to non-specialists in medicine and allied health sciences and to promote statistical literacy within schools

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**Keywords: non-specialist learners; open access learning resources; self-paced learning; statistical knowledgebase; statistical learning; undergraduate medicine**

## Introduction

In this article, I provide a brief summary of my BCME 9 workshop. The workshop involved two linked sessions. The first involved a 20-minute briefing, during which I provided an update on the project *Statistics in Medicine: A risky business?* which was previously funded by the Higher Education Academy (HEA). This enabled me to provide a more general insight into my open access statistics resources for non-specialist learners from medicine and allied health sciences. These resources are hosted by University of Edinburgh (UoE) Medical School on my open access WordPress site, StatsforMedics (*https://medstats.mvm.ed.ac.uk/*). I had designed the briefing session to equip participants for the second component of the workshop, involving a 40-minute interactive breakout session for delegates to provide feedback on nine user cases for my statistics resources. This feedback was gathered through presenting each of the user cases on laminated placemats and providing sticky notes to allow delegates to attach their recommendations to each placemat. Pre-workshop materials were made available via StatsforMedics.

StatsforMedics was released as an open access site in August 2016, which marked the start of a new academic year for medical students at the UoE. I have designed this site for presenting and further developing my longstanding statistics knowledgebase resources. These resources were originally located on intranet pages within the virtual learning environment EEMeC (Edinburgh Electronic Medical Curriculum) for UoE undergraduate medical students. While I have primarily designed StatsforMedics with the intention of supporting undergraduate medical students who are considering use of statistics for short-term curricular research projects, this is with the understanding that students from allied health sciences may also benefit from engaging with the site and its sister site, Statistics CALs. I am currently exploring use of selected content for outreach work in pre-university sectors. Therefore, I would greatly welcome receiving expressions of interest for collaboration from primary and secondary school teachers in addition to higher education professionals.

## Statistics CALs

The sister site, Statistics CALs, offers three self-paced learning resources, each managed with a user interface page that segregates content into chapters. The chapters are provided for ease of integration of statistical topics with clinical teaching across the years of an educational curriculum. The CALs (Computer-Assisted Learning objects) are designed to enhance conceptual understanding of statistics and promote sound interpretation of risk estimates for clinical decision making. The CAL themes are *Summary statistics, Normal distributions and the confidence interval for the sample mean* (CAL 1), *Testing for a difference between two populations* (CAL 2) and *Risk*

(CAL 3). One of the key objectives of the Risk CAL is to empower students to make the right choices from the range of available risk notions in medicine for clinical decision making.

The three CALs feature a variety of approaches to engagement and flexible learning. This includes sensitivity to the learning needs of students with Specific Learning Difficulties. The links to these CALs and to papers explaining the pedagogical rationale for the CAL development and offering a more comprehensive account of CAL content, are available via the menu item SISTER CALS on the top menu bar of StatsforMedics. Among the many icons that are used for signposting content type within the CALs, the icon *Medical statistics in practice – getting engaged with your data!* signposts opportunities for linking theoretical content from a CAL with structured advice about how to perform further calculations and analyse data based on the theoretical content. The use of this icon is illustrated in Figure 1.

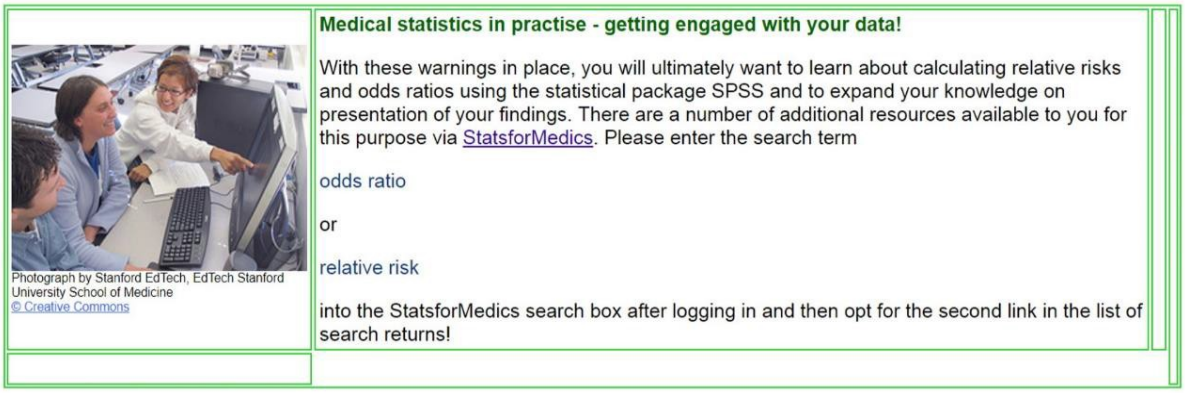



**Figure 1: Use of the Statistics CAL icon functionality to allow better linkage of theory and practice with the estimation of statistical risk**

Further feedback exercises

***Recognition of the student voice: Student engagement through partnership***

The Year 5 Student Selected Component peer-assisted learning module (SSC5b) within the UoE medical curriculum provided an opportunity for me to supervise an industrious student, Kyle H Bennett, in the provision of a comprehensive review of StatsforMedics involving recommendations for making the site content more user-friendly. Kyle also developed an introductory navigational video for StatsforMedics. This video is hosted on the Welcome page, which also serves as the landing page, for the site.

***Higher Education Academy workshop***

In July 2017, I had the privilege of delivering a pre-run of my BCME workshop at the *Annual HEA Conference 2017 (Generation TEF: Teaching in the spotlight).* The feedback I obtained from workshop participants provided a firm foundation for improving interaction with students who wish to use StatsforMedics. Recent developments that I have authored include: creation of the Twitter account @statsformedics, including the option to login via Twitter and provision of a student suggestions form. I offer more details of these developments and those inspired by the SSC5b review in a more comprehensive paper that is currently being reviewed for publication in an academic journal.

Further, to completing my engagement with the HEA workshop feedback, I intend to progress to the corresponding feedback from the BCME workshop. As this feedback has been gleaned from the secondary education sector in particular, it is likely to introduce fresh ideas not considered by the HEA workshop participants. An early observation is the more professional approach to student engagement from the secondary education sector. This has been exemplified by a preference for alternative login access using LinkedIn rather than social media. This conflicts both with the SSC5b student feedback and the HEA workshop suggestions. As an initial response, I have introduced additional functionality to login via LinkedIn. Further discussions with the secondary education sector may lead to teachers instructing students on preferred login approaches, although there may be a need to further explore teacher inhibitions about use of social media in education.

## Invitation to come on board

If you would like to participate in the feedback exercise highlighted at the start of this paper, please register with, or logon to, StatsforMedics and then email me (*Margaret.MacDougall@ed.ac.uk*) to obtain the required password to access the feedback questions. Also:

- are there any undergraduate, postgraduate or professional development courses where you think that any of the statistics resources highlighted in this paper may serve as useful learning or assessment tools?
- do you have suggestions for expanding the utility of these resources through forming relevant partnerships to support funding bids?
- would you like an update on the relevant paper in progress for publication in a peer-reviewed journal?

If so, please let me know!

## Acknowledgements

I express my thanks to:

- UoE Information Services for hosting and maintaining StatsforMedics and for providing learning technology input for both CAL development work and user access to the CAL content.
- Kyle H Bennett, UoE undergraduate medical student, for providing a valuable navigational video and a comprehensive review of StatsforMedics as part of his SSC5b project, the latter of which inspired me to create a more student-friendly learning environment.
- the delegates who contributed so generously in providing feedback at my HEA and BCME conference workshops.

# Third Chances in Education

David Martin

*U3A National Subject Advisor for Mathematics and Statistics*

**Keywords: U3A; mathematics; statistics; geragogy; lifelong learning;**

## Introduction

If we are talking about a third chance in education, what might we mean by a first chance and a second chance?

We could argue there are even more chances than the ones I am about to list, as we could for example add others such as pre-school education, but for this I am referring to school as a first chance and Further Education (FE) as a second chance.

Why might we consider FE as a second chance? Having taught for most of my teaching career in different forms of FE I found it a privilege to work with students who had given up at school, learnt to read in prison, re-engaged with education to gain access to university after bringing up a family, or sought to regain their mathematical pathways after a brain injury.

But does lifelong learning end there? We learn as a child at school from other people's curricula. Success or failure in assessment leads on to the next stage of learning. In FE we select from various curriculum choices on offer and success in examinations enhances progression in a career or to a higher level of study. For the elderly who do not need to progress to the next stage of learning or gain success in examinations for the advancement of their career what might be their motivation? For them it is more about social engagement, keeping active mentally and physically, and the fun of learning.

The drive and focus for lifelong learning may change over the years but it engages us at all stages from child to old age and with grandparents acting as informal teachers of their grandchildren, the learner becomes teacher.

The University of the Third Age (U3A, www.u3a.org.uk ) is a self-help learning organisation with members in local groups all over the UK, affiliated to a national organisation, *The Third Age Trust*. It has its origins in 1973 in France with the Université du troisième âge where universities took the lead. In 1982 in Cambridge, UK, the structure chosen for this country was more self-help with independence from universities. Growth has been rapid and now 35 years later, there are over 1000 local groups and over 400 000 members. Who might be a typical member? If there is a modal member, it might be a female with a professional, managerial or administrative background, about 70 years of age. (Beckett, 2014)

We might attach names to the three stages of learning from pedagogy, through the less well-known andragogy to the even less well known geragogy of learning that can take place in the context of reduced cognitive function combined with rich past experience.

## U3A national and local

Some may well be well aware of how the U3As operate but for those who do not:

- The local U3As are autonomous and affiliated to a national body called the Third Age Trust

- The Third Age Trust arranges insurance, draft constitution, recommended guidelines, and provides a magazine, and library of resources
- A typical U3A has a (monthly) main meeting with a speaker and a number of interest groups ranging from Art Appreciation, through Book reading, French Conversation, Gardening, Philosophy to Ten Pin Bowling.
- The interest groups meet in a mixture of members' homes and community rooms.
- Nationwide there are 70 Mathematics and Statistics groups from Maths for Fun to building on A level knowledge
- There is a National Subject Advisor for Mathematics and Statistics
- The National Advisor supports U3As in setting up and maintaining interest groups and writes a termly newsletter and sends out a termly problem sheet

Let me illustrate a mathematics and statistics group by sharing about the local group I lead with a colleague:

- I run a Maths for Fun group that meets monthly for free in the local Fire Station Community Room that provides a display screen, flipchart and kitchen
- With a membership of about 10
- Shared leadership with an engineer with no prior experience of teaching
- Sharing solutions to starter activities we are interested in drawing out that there is:
    - More than one method of solution
    - The value in sharing different approaches
    - The value in use of history to place difficulties in the context of decades, centuries and millennia of development, and to give an engaging context to the work
- Planning sessions involve review of previous sessions, teacher training (on use of discussion, value of engaging with misconceptions, effective questioning,…) and planning of future sessions
- Maths for Fun sessions - Starter activity, History / Use of Mathematics, Refreshments, Mathematical Skill in context

## Case studies

Below are anonymised samples of the enquiries made to the National Subject Advisor in Mathematics and Statistics. What advice would you give?

### *Case study one – about to start a group*

> We are just about to start a maths group - called Maths for Fun. We had our first session and discovered that although some of our students had O level maths they were scared of maths.
>
> So we are hoping that you will help us with a strategy for tackling their problems and getting them to love maths.

### *Case study two - just started a group*

> I have just started a Maths for Grandparents group. The reason for starting the group was the need to speak the same maths language as my grandsons and have a chance of not confusing them with their homework.
>
> If you have any suggestions, we would be pleased to receive them.

### *Case study three – started a group a year ago*

> Having seen an article in the Third Age about Maths for Fun I was persuaded to start such a group. This has been running successfully now for over a year. However, I am beginning to run out of ideas for where to take the group.
>
> I would appreciate any assistance you can offer.

### *Case Study four – started a group two years ago*

> Our Group started 2 years ago. Initially, we were a small group of people, mostly of O level and a little better standard, and the leader took us through the current course requirements with some A level syllabus. We are a little too small to split into an advanced and a medium group, but we worry about the content of our studies to maintain the interests of such a wide compass of abilities.
>
> Do you have any helpful suggestions?

## Reference

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# Supporting students preparing for university admission tests and STEP

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**Keywords: STEP; university admission tests; University of Cambridge; A level; further mathematics; higher education**

## Introduction to university admission tests

There are a wide variety of university admission tests now used by universities as part of their admissions procedures.

***For mathematics degrees:***

- Test of Mathematics for University Admission (TMUA). Accepted by a number of universities as part of a 'reduced offer'.
- Mathematics Admissions Test (MAT). Used by Oxford and Imperial College London, and also used by Warwick as part of a 'reduced offer'.
- Sixth Term Examination Paper (STEP). Used by Cambridge since 1987. Several universities accept STEP for a 'reduced offer', and Imperial College London asks for STEP grades if an applicant has not sat MAT.

***For some other mathematically related degrees:***

- The University of Cambridge sets admission tests for Engineering (ENGAA), Natural Sciences (NSAA) and Economics (ECAA). Some applicants for Engineering are also asked for a STEP grade.
- The University of Oxford asks Physics applicants to sit the PAT.

Details about all of these tests can be found on the Cambridge Assessment Admissions Testing website.

## What is STEP?

STEP is the 'odd one out' in the admissions test spectrum, as it is the only one taken at the end of Year 13 (the other test are taken in October/November of Year 13).

There are three STEPs in mathematics, STEP I, STEP II and STEP III. Currently, each STEP paper contains 13 questions: eight pure mathematics, three mechanics, and two statistics and probability. Students are asked to attempt six questions during the three hour examinations. The results are graded as S (highest grade), 1, 2, 3 and U. Grade boundaries can be found in the examiners' reports available on the Cambridge Assessment Admission Testing website.

Offers to read mathematics at the University of Cambridge almost always contain STEP as part of the conditional offer, usually grade 1 in STEP II and STEP III but there may be some variation with grade S and grade 2 offers.

## Why do universities like STEP?

There are a few reasons why universities like students to study for STEP. The questions are of a similar style to university mathematics, and they require more advanced

problem solving skills than A level mathematics. The questions are designed to take longer than in A level, so there is an expectation that students will get stuck, have to spend some time thinking and might have to try a few different approaches before solving the problem.

STEP also tests motivation – (very) few people can do well on STEP without putting in a considerable amount of time working through past papers. It is supposed to test for mathematical 'ability' rather than how well a candidate has been taught, but being familiar with the style of question and experience of 'being stuck' are very important for success.

In 2017 (DfE) there were over 4400 students achieving A* in A level Further Mathematics compared to approximately 250 places to read mathematics at Cambridge every year. The University of Cambridge uses STEP to distinguish between these high attaining students.

The increased interest in STEP from universities other than Cambridge, as well as an increased awareness from students had led to a remarkable increase in the numbers taking STEPs over the past ten years as shown in figure 1.

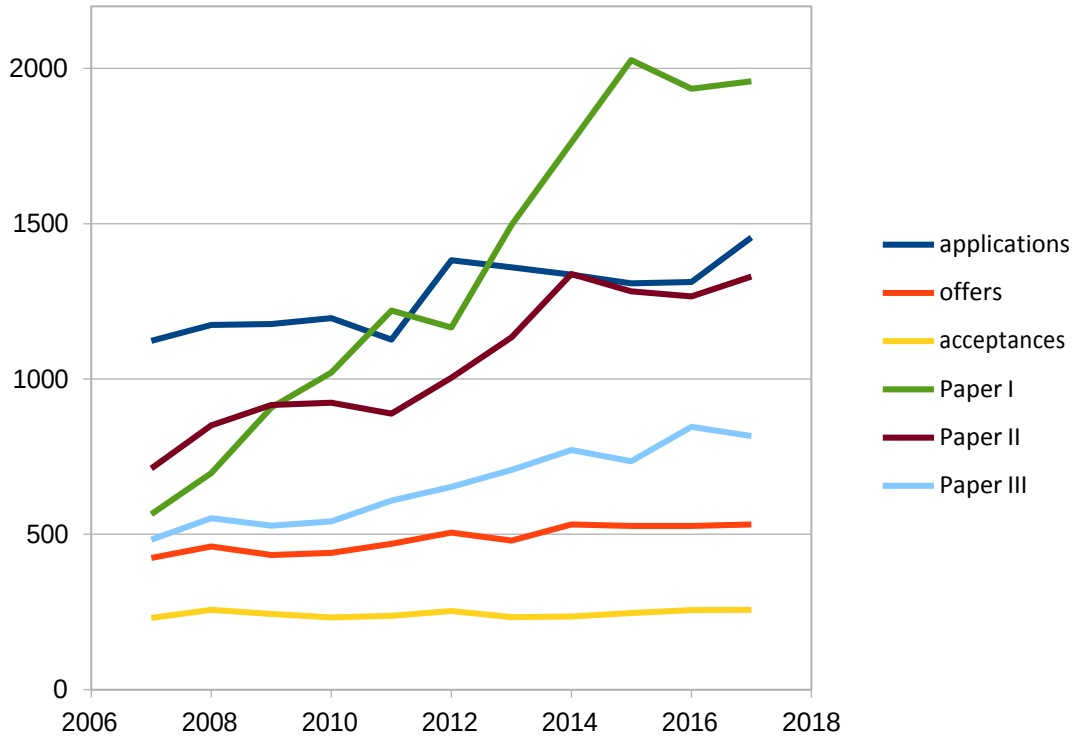



Figure 1: Entries for STEP I, II and III as well as applications, offers and acceptances to read mathematics at the University of Cambridge. (Data from the STEP Coordinator at the University of Cambridge)

## What are the challenges facing schools and students?

STEP questions are hard, much harder than A level questions and students who have not had much experience of being stuck on mathematics problems before can find this very discouraging. Some of the other challenges are:

- Schools not having much experience of students sitting STEP
- Teachers not being able to support students (time and/or difficulty constraints)

- Students who study A level Mathematics over two years are not able to access many of the questions (even STEP I) until just before the exams
- Students may be the only student at their school sitting STEP, which can be a lonely and demoralising experience
- Lack of student confidence

**What is the STEP support programme?**

The STEP support programme is funded by the University of Cambridge and replaces the STEP Easter school, which could not be offered to all state school students and was possibly too late in the year to prepare them for sitting STEP in June. The STEP support programme resources are freely available to everyone. The programme aims to address the challenges faced by schools and students when preparing for STEP. It does so in the following ways:

***Assignments***

There are two main types of assignments. The 'foundation assignments' are designed to slowly increase student confidence in answering STEP questions, and also introduce them to some new (and hopefully interesting) areas of mathematics. Each assignment has a carefully selected STEP question which is preceded by some preparation, practising the skills needed to answer the STEP question. These assignments are intended for use from January of Year 12 onwards, and the first few can be answered using just GCSE knowledge. They prepare students for STEP I, or can be used to improve students' problem solving abilities before tackling the STEP II and STEP III assignments. Another feature of the foundation assignments is that they introduce (and justify) many of the results in C3 and C4, such as integration techniques and trigonometric identities, so that the students can access a greater range of STEP questions earlier. Solutions to all the assignments are available on the website.

The STEP II and STEP III assignments have a different structure. Each has a selection of STEP questions (with no preparation), a hints document, a solutions document and topic notes. These are intended to be used in Year 13, and they assume that students have worked through the foundation assignments first.

***The discussion forum***

The discussion forum is the only aspect of the site that users have to register to use (although the comments are visible to everyone). Here students (and others!) can ask questions about the assignments on the website, or any other STEP question, or any aspect of STEP and sitting the examinations. The forum is monitored by staff and students at the University of Cambridge.

***State school offer-holders days***

The on-line programme is supplemented by a day in Cambridge discussing STEP techniques and working through STEP questions with other offer-holders. The chance to attend one of these days is offered to all students who attend a UK state school and who have an offer from the University of Cambridge involving STEP II and/or STEP III. The day consists of a combination of talks and workshops where offer-holders can meet current Cambridge mathematics students. This year over 200 offer-holders came

to these events. The materials from these days can be found at https://maths.org/step/materials-march-days-2018.

### *Specification changes for 2019*

The ordering of the assignments, and the assignments themselves, will be changed in the second half of 2018 to reflect the changes in the STEP specification, which have arisen due to the changes in A level. Some details of the planned changes can be found on the Cambridge Assessment Admission Testing website (specifically http://www.admissionstesting.org/for-test-takers/step/about-step/), and the new specification will be available in the summer of 2018.

### Other ways students can prepare

The main way most students like to prepare is by working through past papers. These can be found on the Cambridge Assessment Admissions Testing website along with examiners reports' and partial solution documents. There is also a STEP database which holds all the questions from 1987 that can be searched by keyword. There are hand-written solutions to all STEP papers from 1996-2017 available from MEI.

In the early stages of preparing (Year 12, and the start of Year 13), students could try some MAT or TMUA papers, for which the specification is essentially the same as for AS mathematics.

The BMAT and other Cambridge and Oxford admission tests (such as PAT, NSAA, ENGAA, PBSAA) are another good source of interesting mathematics questions. Many of these only rely on GCSE mathematics knowledge. Details of these tests can also be found on the Cambridge Assessment Admissions Testing website.

### Website references

A level results: https://www.gov.uk/government/statistics/a-level-and-other-16-to-18-results-2016-to-2017-revised
Cambridge Assessment Admissions Testing: http://www.admissionstesting.org/
STEP database: http://stepdatabase.maths.org/index.html
Solutions to STEP questions from 1996-2017: http://mei.org.uk/step-aea-solutions
The STEP support programme: www.maths.org/step

# Lesson study, professional development and the development of mathematical and pedagogical understanding.

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**Keywords: lesson study (LS); professional development (PD); mathematical concepts; pedagogical concepts.**

## Introduction

Over the past five years or so, we have been involved in delivering professional development courses to inservice teachers through our local Maths Hub. The research discussed here, has developed through those experiences. These year-long professional development courses are based around a programme of five PD days with expected activities back in participants' schools between each of the days, mainly involving a form of lesson study. We reflect on this experience to draw parallels between how people learn mathematics (how they develop mathematical concepts), and how people learn about teaching and learning mathematics (how they develop pedagogical concepts), and use those parallels to deepen our understanding of both. As part of our discussion, we will look at a particular piece of dialogue, to illustrate and develop those ideas.

Here we provide a brief overview of our current thinking, targeted at the BCME audience. A full discussion of our research will be the subject of a longer paper.

## Developing mathematical concepts

We claim that the essential elements for the meaningful learning of mathematics in classrooms are:

- Genuine problems
- Active role for students
- Connecting to experience (physical/visual/social/mathematical experience)
- Connecting mathematics/generalisation
- Social dialogue and reasoning

Where do these concepts come from? Firstly they come from a theoretical perspective; they are effectively a translation of Vygotsky's theory of scientific concepts (Vygotsky, 1987) into pedagogical elements (Swanson, 2013). Alternatively, they could be seen to come from practice, or empirical research, (including research directly influenced by Vygotsky's ideas). They are the common sense of much of mathematics education research; the ideas seen in the National Council of Teachers of Mathematics **(**NCTM) in the US, and elsewhere, representing a mix of social constructivism, realistic mathematics education (RME), connectionism and other perspectives.

We add that connecting mathematics/generalisation is the least developed in both traditional teaching (where topics are atomised) and in more meaningful teaching (such as problem solving lessons where different models or methods are shared, there is no necessary generalisation through or between those approaches for students). To achieve this a new problem is often needed which pushes those methods or models

together e.g. by asking questions such as, *Where is that 6 in this solution? What is the same/different? Which method is most efficient?*

## Developing pedagogical concepts

The next leap we want to make is to say that these elements, seen as essential for learning mathematics, should have parallels for learning about teaching mathematics.

Why? In Vygotskian terms, with concepts about pedagogy we are still talking about conscious concepts which appear in systems, so, scientific concepts. Also we are still at a high level of generality (our scheme so far is pretty simplistic) so should be expected to apply to any system of scientific concepts. We would argue that the essential elements in learning to teach mathematics therefore are:

- Genuine problems
- Active role for teachers
- Connecting to experience (physical/visual/social/teaching experience)
- Connecting pedagogy/generalisation
- Social dialogue and reasoning

If we take these parallels between learning mathematics and learning the teaching of mathematics as our starting point, we can then look at similarities and differences to deepen our understanding of each.

## Lesson study

If we examine LS as a form of PD we see that it satisfies many of the criteria above.

- Genuine problem - something teachers are trying to understand (the research focus), or the task of designing and teaching a lesson.
- Active role - teachers have to go and do the LS. They are not just being told about what might be good, they are actively trying it out.
- Connections through experience - any ideas about pedagogy in the PD materials or planning are enacted in the classroom, so there is a rich concrete experience of big ideas in practice.
- Social- LS is collaborative, involves dialogue and reasoning in planning and analysing

As with mathematics, in developing pedagogy through LS, connections/generalisations is perhaps the least likely to occur/hardest to generate. It may happen that in analysing the lesson wider connections will be made, but that is not inevitable. It would also be possible to just stay stuck in the concrete of how to make this specific lesson better, or what to do next with the particular class.

There are different ways to encourage these connections/generalisations. In the Japanese LS model there is a 'koshi' (knowledgeable other) who will try to generalise at the end of a LS analysis session (Takahashi, 2014). Or similarly if the LS is written up, this may force generalisation and wider connections to be made. We find that in our system and experience, it is in the PD sessions that these generalisations and connections can become more likely, or deliberately pushed for. At these sessions we have pairs who have done their own LS who then come together with others who have done similarly. Their experiences are sufficiently similar and sufficiently different to raise teachers out of their own particular experiences and encourage the making of connections between the different experiences. One of the key features of PD, for us, is to deliberately push for these generalisations, by posing new questions or problems that force the different experiences together. So questions like, *What's the same or what's*

*different about these experiences? What would you do in those circumstances? Is that always the case?,* and even explicitly, *What is general about that experience?*

We think this overall combination of LS and PD leads to more conscious and systematic thinking. Thinking which is more conscious and systematic can be more easily tested and developed further.

## The transcript

We offer an example of the sort of talk that can arise in these PD sessions. The full transcript is around 600 words long, so we are sharing an excerpt. It is mostly call and response between teacher A and three other teachers (B, C and D). Here it was stimulated by the introduction of a lesson that they were all going to go and teach as part of a LS.

| | | |
|---|---|---|
| 1 | A | I started with that [*What else do you know?*] and then I don't know what to, at what point, do I dwell and say, *Well can we come up with a strategy, should we look for a lowest common multiple?* Do you do that or do you just keep letting them do it until they sort of discover that for themselves, how long would you spend on it? |

The teacher starts by asking the above question: he is considering, how do we push students toward a strategy, and how do we bring in mathematical connections. Arguably this is a questioning of pure constructivism, with an alternative of either more guided learning (or just transmission). In terms of the elements of our schema we see an initial active role for students (*What else do you know?)* followed by a desire to emphasise the connections in mathematics, and a form of social learning (if we take the teacher as the social element). Other teachers respond:

| | | |
|---|---|---|
| 2 | B | …we do start with, "Do whatever you can". The majority of students then after a while start developing their own strategies… a year 7 girl yesterday… she's working out 23%. So she said, "Well, I can work out 10%, I can work out 20, I can work out 3, I don't know when to stop"! That was her question. So she wasn't looking forward. And so I insisted - well carry on until you think "Oh I'm getting there". And I think by asking those kinds of questions then they'll naturally start developing it. |
| | | [And] |
| 5 | C | We do similar things with how we work out percentage currently, in year 6 especially, and very much the children all go their different ways, and we don't go, "This is the most efficient"… |

These statements argue that student needs an active role in constructing their own way of solving it, in order to own it and be able to use their understanding in different circumstances later. In terms of our system of pedagogic concepts, asking what else do you know is a conscious pedagogic strategy for B. Also (for B and C) there is a conscious strategy of letting students keep going until they see how they can end it. For A we had the bringing in of connections. And these seem to be in contradiction - they both seem important but seem to lead to different practice.

Later in the conversation…

| | | |
|---|---|---|
| 8 | A | I'm conscious of not being too prescriptive, not giving them a strategy, but I also want those links that are there in the curriculum to be made, so at what point, if they bring it up, can you discuss it? |
| 9 | D | I think the minute that you try to start to formalise it, it becomes something else that you've got to remember, or something they feel they've got to remember, and they're like "oh how do I do this again" |

| | | |
|---|---|---|
| 10 | A | Yeah absolutely… I'm just conscious of the fact that if you miss opportunities to… |
| | | [And] |
| 13 | B | See I think you open up to the opportunities but accept that not everyone on that day is ready for it, and accept that what they're doing is right for the journey they're on. There'll be another opportunity next time, and another one, and another one…it's a long journey. |

Here another teacher (D) comes in, and this feels an even more general perspective. They are making connections to the wider ideas of rote learning and memory, building on what has come before in the discussion, to argue why it is important that students construct their own knowledge and do not just memorise someone else's. Meanwhile A is at pains to say he agrees with that perspective, yet his problem has not been resolved. What seems important is that he is working through these pedagogical ideas at quite a conscious level. We would argue that such discussions are vital in developing conscious pedagogical understanding (even if it is impossible to show development of any participant's ideas within the conversation).

In the dialogue above it is the contradictions between the elements of our schema (individual versus social, active role versus connections) that seem to sustain the discussion. A's final contradiction in the discussion, which seems important, is between what B, C and D argue for students in mathematics and how they argue with A about pedagogy. They do not let him find his own path, or follow his own journey to pedagogical understanding but actively try to convince him (in our schema, social dialogue and reasoning). Does this imply a difference between learning pedagogy and learning mathematics or does it prompt further questioning of B, C and D's approach?

## Conclusion

LS has many of the features we would want to help develop pedagogic understanding. Longer term PD courses that use LS, and then bring the teachers who have done it back together, can lead to much higher level discussions, where connections and generalisations are made, particularly if PD leads are conscious of their role in encouraging that. Also we feel there are clear parallels between how we learn mathematics and how we learn about teaching mathematics, but also there are differences. We think using this framework, and looking at the rich discussions that take place in these PD sessions, can begin to help us deepen our understanding of both.

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# Teaching Problem-solving in the Primary Classroom: the ITE Perspective

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**Keywords: problem-solving; primary mathematics; initial teacher education (ITE)**

## Introduction

The purpose of this paper is to discuss how we as Initial Teacher Education (ITE) providers develop our undergraduate student teachers' understanding of a problem-solving approach to the teaching of mathematics in the primary classroom and how we enable them to have the confidence to develop their pupils' problem-solving skills. We discuss the challenges experienced by us as tutors and by our students as non-specialist student teachers during teaching sessions at the university and during school experience. We also discuss the strategies we use to overcome these challenges. We collected data from final year undergraduate student teachers on the BA (Hons) in Primary Education programme after they completed their final teaching experience in school. The data were collected in the form of a questionnaire that focused on problem-solving approaches seen in schools and student teachers' confidence in teaching through a problem-solving approach.

## A problem-solving approach

Problem-solving is at the heart of mathematics and is something that we as humans engage in on a daily basis, often without thinking. According to Ollerton (2011: 90) "the innate need to solve problems is the fundamental basis for the existence and the 0construction of mathematics". One of the key aims of the current National Curriculum (Department for Education 2013: 3) requires pupils to "solve problems by applying their mathematics to a variety of routine and non-routine problems with increasing sophistication". This should involve more than just adding a problem on at the end of a topic or having a 'problem-solving Friday' for pupils to apply what they have learnt during the week. Ollerton (2011: 90) uses the phrase "problem-solving pedagogy" to describe teaching through a problem-solving approach. He stresses the importance of learning over teaching and acknowledges the role of the teacher as one of facilitator who enables pupils to fully engage with tasks in their own way. This is the role that we as mathematics education tutors seek to develop in our student teachers and below is an outline of some of the challenges we face in the process.

## The challenges facing ITE providers

Teaching through a problem-solving approach involves a significant shift in mindset for some of our student teachers. If they have not experienced this pedagogy in their own mathematics learning, then it is difficult for them not only to understand what this would look like in the classroom but also why we should even teach mathematics in this way. For many, the emphasis was on learning rules and formulas rather than on conceptual understanding. It is not unusual to hear comments such as, "Why shouldn't we only teach rules? I learnt it that way and it didn't do me any harm." from some of our students in the first year of the programme. This lack of conceptual understanding leads to gaps in subject knowledge that we have to address alongside developing

pedagogical subject knowledge. Exposing gaps in subject knowledge also reveals the anxieties experienced by some of our student teachers when their conceptual understanding is challenged or when they engage in problem-solving tasks.

Further challenges are faced when a problem-solving approach is not seen by many of our student teachers during their time in placement schools. There are many reasons for this, one of them being the mentor's own approach to or understanding of problem-solving that may be due to their inexperience as a teacher or gaps in their subject knowledge. An increasing number of our partnership schools are investing in different text book schemes to deliver mathematics throughout the school. Appropriate training in and/or understanding the principles behind these schemes are not always in place to enable their effective use. As a result our student teachers are not always exposed to the best pedagogical models of mathematics teaching.

Schools are accountable to parents and to Ofsted for the progress made by pupils and this is reflected in the National Curriculum Test (NCT) data and Performance Tables that are published nationally and open to public scrutiny. Teachers have to provide evidence to support this progress which may not always be possible after every problem-solving activity. Developing pupils' skills and understanding through problem-solving is risky and time-consuming. It is a long-term investment that a teacher has to make before they see progress and it is not always measurable. The pace and depth of learning will be different for individual pupils. Differentiation will be more focused on learning outcomes and starting points rather than on having a number of different activities, as described by Ollerton (2011). The challenge we as mathematics education tutors and ITE providers are faced with is how to prepare our student teachers to teach mathematics through a problem-solving approach in the current classroom climate.

## Strategies we use

Tutors use a range of strategies to try to overcome these challenges and to develop not only student teachers' understanding of what this approach involves but also their confidence in applying it in their primary classrooms. We have outlined some of these strategies below.

### *Modelling*

During seminars, tutors model a problem-solving approach and share our Group Problem-solving Rules with students so that our expectations of their involvement are made explicit. These are:

- You are responsible for your own work and behavior
- You must be willing to help any group member who asks
- You may only ask the teacher for help if everyone in the group has the same question.

We provide tasks that challenge our students' thinking so they go through the same processes experienced by pupils in class. This shared experience is important not only for developing their own problem-solving skills but also to develop their understanding of how to support their pupils to be resilient and to find ways to work through problems. We ask questions such as, *How did your group get started? What caused some difficulty? Which did you find most helpful? Did any group do it a different way? What might you do differently next time?*

The use of questions like these help students to notice what is happening and to share their experiences by marking and re-marking as Mason (2002) suggests. With continued exposure to question such as these, they become internalised so that student teachers can in turn use them with their pupils.

One of the activities we use with students is to give picture symbols to represent the numbers in the multiplication facts for seven, e.g.  ×  = , and ask them to order them correctly. This involves students engaging in discussion and reasoning to work out what number each symbol represents before arranging them in order. Throughout the activity we encourage engagement, use of resources and discussion. Afterwards students share their solution strategies, what they may have noticed about the task and their thought processes. We ask them to discuss how the experience might inform their role in supporting the development of pupils' mathematical thinking.

### *Video analysis*

The use of video analysis is an effective way of engaging students in discussions about teaching and learning in the classroom. Mason (2002) discussed the importance of noticing what is happening in one's practice and of marking what is noticed so that it can be acted upon later. Student teachers have little or no experience of this type of noticing, so the use of video analysis supports this development. However, our experience of using footage of whole-class lessons is that students focus on the classroom environment and how the teacher is managing the behaviour rather than on the teaching and learning that is taking place. Amador (2017) had similar experiences when using video simulations with her pre-service teachers. She found that providing opportunities to work collaboratively to plan teacher and pupil responses in a Video Simulation Task enabled the pre-service teachers to develop a better understanding of teaching and learning in the mathematics classroom. We have been able to address this to some degree by using footage of one-to-one teacher interventions or small group activities rather than examples of whole-class teaching. We found that this enables students to become more disciplined in their noticing (Mason 2002) and to look beyond what they see on the surface so that they can begin to notice more effectively in their own practice.

### *School experience observations*

When student teachers are in school we, as link tutors, observe lessons and provide feedback in the form of a post-lesson conversation in which student teachers are encouraged to discuss positive aspects of the lesson and areas for further development. We provide them with a narrative of the lesson with strengths and targets linked to the Teachers' Standards. Comments made by tutors focus on the use of resources, rich mathematical tasks, questioning to deepen understanding and opportunities for reasoning. Through the use of some of these anonymised, real lesson observation narratives and targets, tutors' feedback is analysed in sessions to stimulate discussion of a problem-solving approach. This helps student teachers to develop a clearer understanding of how to use a problem-solving approach in their practice and they begin to experience the positive outcomes this can have with their pupils.

### *Theory and practice*

It is important that our student teachers have a clear understanding of how theory underpins all aspects of their pedagogical practice. We introduce them to theories of

learning from the works of Bruner, Piaget and Vygotsky so they develop an understanding of theory and its impact on practice in the classroom. More recent work by Askew, Haylock, Mason and Thompson, is used with student teachers so that they understand how theories can be developed with further research and the importance of updating their pedagogical subject knowledge. We explore the use of questioning to develop reasoning in the classroom and the related research and how effective questioning is linked with the use of rich mathematical tasks. Students have opportunities in seminars to discuss journal articles related to mathematical topics so they learn how to critique articles and how to use them to build an argument particularly when writing their assignments.

### *Reasoning module*

Student teachers in their final year of the BA programme take a module that focuses specifically on mathematical reasoning. They explore current guidance on mathematical problem-solving with a particular focus on reasoning skills. They engage with a variety of tasks and strategies that are effective in exposing children's ability to reason mathematically. Students also explore how these tasks can be used in the classroom, with a view to using them to develop mathematical reasoning skills.

On returning from their final placement, student teachers were asked to complete a questionnaire about their experiences of problem-solving in their schools. The majority of respondents had a clear understanding of what a problem-solving approach involved but very few experienced this in their placement school. In some of the schools problem-solving was taught through NCT items and in others it was taught at the end of a topic. Most of the student teachers who responded believed they were gaining confidence with regard to teaching problem-solving and most also wanted to see more examples of this approach in schools.

## Summary

In this paper we have discussed how, as ITE providers, we develop our student teachers' understanding of teaching mathematics through a problem-solving approach. We have discussed some of the challenges we and our undergraduate students face in terms of subject knowledge as non-specialists and their experiences in school. We have shared some of the strategies we use in our teaching sessions to develop understanding of and confidence to teach this approach. We will continue to develop a problem-solving approach with our students and to work with our partnership schools so that our student teachers' university and school experiences are complementary.

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# Connected A Level Programmes of Study

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**Keywords: A level; connections; dependency; programme of study**

## Introduction

Programmes of Study (PoS) tend to work through specifications topic by topic. Can we improve on this to develop mathematical thinking by looking at the connections in the material and travelling through the mathematics in a different way? The Further Mathematics Support Programme (FMSP) Wales has mapped prior and dependent topics for every topic in our Scheme of Work (SoW) for the new Mathematics (M) and Further Mathematics (FM) A level syllabi. Can we use this resource to grow our students' (and teachers') understanding of the patterns running through mathematics?

## FMSP Wales Scheme of Work Connections

Mathematical connection is embedded into our SoW. We have highlighted this through the connections afforded by dependency between topics (see figure 2) and through our requirement for each topic to include:

- bigger picture (history of mathematics, connections to other topics, ideas underpinning mathematical concepts that may go beyond the syllabus, beauty, etc.)
- 'real-world' and workplace applications and connections
- how the topics are reflected in STEM (and subjects using statistics) undergraduate studies
- questions drawing on other topics

Key references include:

- avoiding "…rapid, superficial coverage of a large number of topics each year." (Englemann et al., 2001: 292)
- "maintaining close connections between tasks and children's mathematical thinking" (Clements and Sarama, 2004: 82)
- procedural fluency, relational understanding and schematic understanding, so the learner has a sense of how mathematical actions, processes and objects are connected. Such understanding leads to the learner developing a mental image of mathematical connectivity (Cambridge Mathematics Framework, 2015)
- connectivity: making important connections explicit in a consistent way that helps these connections to be referenced more easily, including those that may span multiple areas or otherwise tend to escape attention in existing curricula (Cambridge Mathematics Framework, 2018)

To write topic dependencies, we started with the specification to form expanded learning objectives (LOs). Next we identified the prior learning required and dependent topics. Once all the topics were completed, we put them on a spreadsheet and added each topic as a dependent topic for their prior topics and vice versa. We ended up with a summary for every topic, e.g. figure 1 for quadratics

| | | |
|---|---|---|
| Quadratics GCSE (general intro.)<br>Multiply brackets (setting up completing the square)<br>Simultaneous equations (general intro.)<br>M2.1.2a Indices & Surds (exact roots) | Identify a quadratic function<br>Factorise a quadratic with coefficient of x2 =1<br>Factorise a quadratic with coefficient of x2 ≠1<br>Derive the Quadratic formula<br>Use the quadratic formula to solve quadratic equations<br>Calculate the discriminant of a quadratic function<br>Understand the conditions for real, repeated and no real roots.<br>Recognise the nature of roots graphically<br>Complete the square.<br>Use completing the square to solve quadratic equations.<br>Use completing the square to find the maximum or minimum value of a quadratic function & the line of symmetry.<br>Solve simultaneous equations in two variables by elimination and by substitution, including one linear and one quadratic equation.<br>Find the points of intersection or the point of contact of a line and a curve.<br>Make sketches relating to the above.<br>Use Geogebra to investigate the above. | M2.1.2c Inequalities (Quadratic inequalities)<br>M2.1.2d Polynomials (manipulation skills, simple polynomials)<br>M2.1.2e Graphs & Graph Transformations (important function)<br>M2.1.3b Geometry of circles (completing the square, solving line & circle)<br>M2.1.5 Trigonometry (trig. quadratics)<br>M2.1.7d Stationary Points (solve quadratics)<br>M2.2.7b Constant acceleration formulæ (solving s=ut+½at2)<br>M2.3.2a Simplify Rational Expressions (factorising expressions)<br>M2.3.2f Functions in Modelling (quadratic models)<br>M2.3.5c Compound angle formulae (quadratics)<br>M2.3.7 Integration (find intersection points of curves)<br>FM 2.1.2 Complex Numbers (solving quadratics)<br>FM 2.1.4a Polynomials (identity: comparing coefficients; relationship between coefficients & roots)<br>FM2.3.1a Momentum and Impulse (simultaneous equations)<br>FM 2.4.8 Differential equations (auxiliary equation) |

Figure 1. Quadratic dependencies from FMSP SoW

## The Master Mind Map

Having completed the dependencies, we illustrated them on a Mind Map. This formed a beautiful diagram (figure 2), although it is difficult to read unless printed at A0 size!

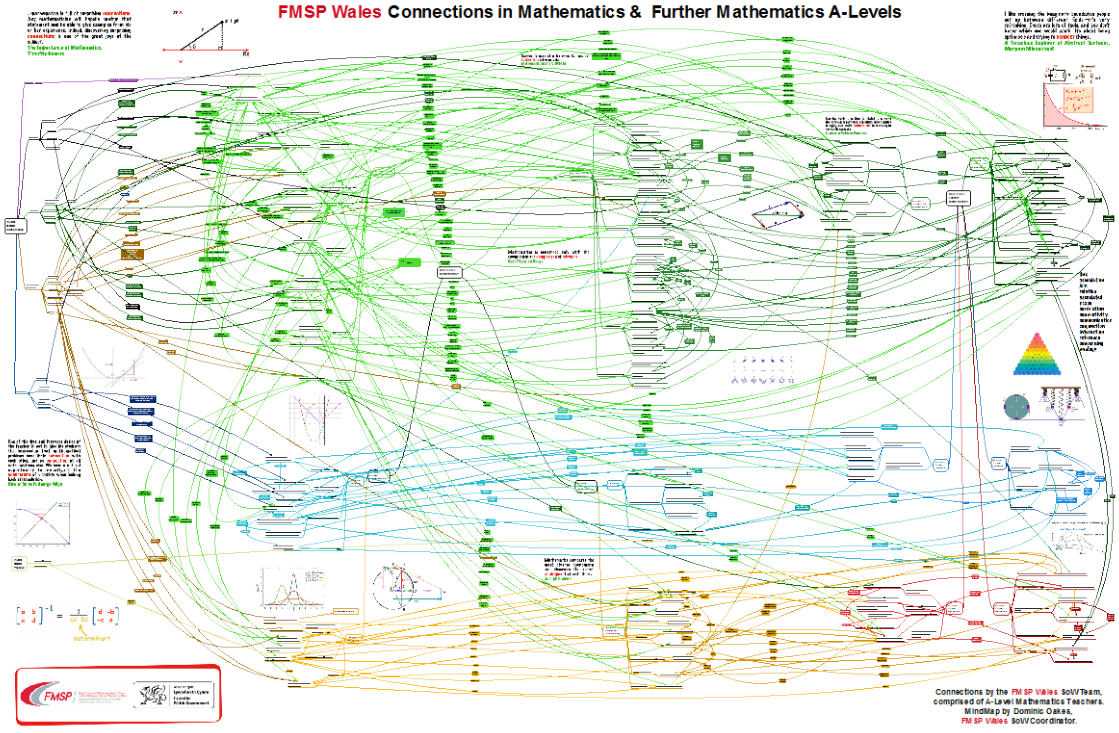



Figure 2. FMSP Mathematics and Further Mathematics Mind Map. The map shows all Specification Topics mapped from their papers and all the dependency connections. Green is pure, blue is statistics and orange/red is mechanics.

We can use this Mind Map to generate maps showing content connections. For example, for quadratic equations we have figure 3:

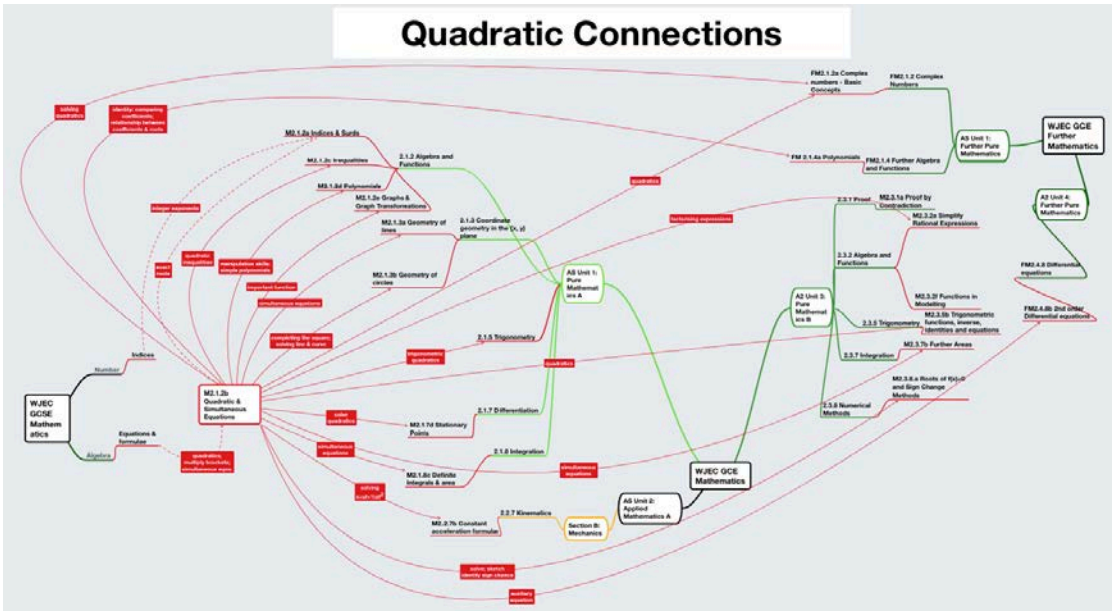



Figure 3. Quadratic equations in FMSP Mathematics and Further Mathematics

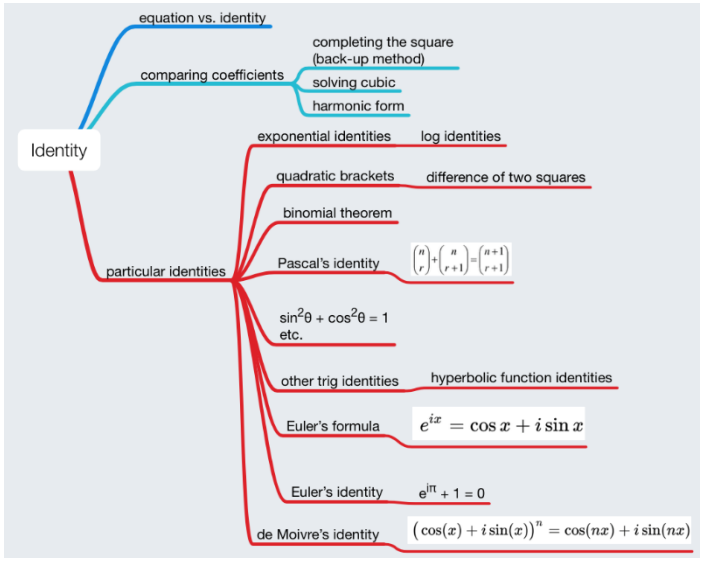



Figure 4. Identity

We are looking at ways to develop our map for wider functions such as clickability to resources. We also intend to add 'conceptual' functionality so that content using the same mathematical ideas can be identified (see figure 4).

## BCME9 Workshop

Helen Hayes (HH, from the FMSP SoW team) spoke of her experience of working on the SoW. She mentioned the following as helpful to her planning:

- LOs helped understanding of the syllabus: discerning exactly what WJEC wanted, looked at similar strands from M AS to FM A2 – clarifying progression
- prior learning and dependencies: increased understanding of links from M to FM; connection to GCSE/Additional Mathematics; link the additional mathematics work to the new A level syllabus
- small topics impacted by/on a lot of others: not necessarily in the same area of mathematics; then linked forward – continuum
- connectivity and continuum: continuum and building blocks; I Like!; never having to unlearn things to learn a new model; can always go back to basics
- aim for students to make the connections: applying past knowledge; the "Oh, it's like …" and "we could use …" moments

- connections at A level filtered down: having seen the connections at A level, connections lower down the school became more obvious – I was looking for connections!; more connections (anchor points?) emphasised at KS3 and KS4
- team meetings to pull SoW together: process; meeting together to bring it all together – teams – discussing with others; working with others (FM can be lonely – only teacher at my school); creating something with a purpose.

Having introduced the idea of connection and the benefits of using them to design PoS, participants made topic maps (see figure 5). Whilst doing so, we discussed the possible impact of keeping connections central to our thinking regarding curriculum design. There were a number of interesting points from participants:

> P1: Can see how using connection will lead to more 'aha' moments and enthuse children
>
> P2: Difficulty of weaker students not being able to cope with multiple approaches – essentially, they can only cope with algorithms
>
> P3: As a tutor the 'standard approach' has generally not worked (which is why the student is coming for tutoring) so have to look for different ways into topics

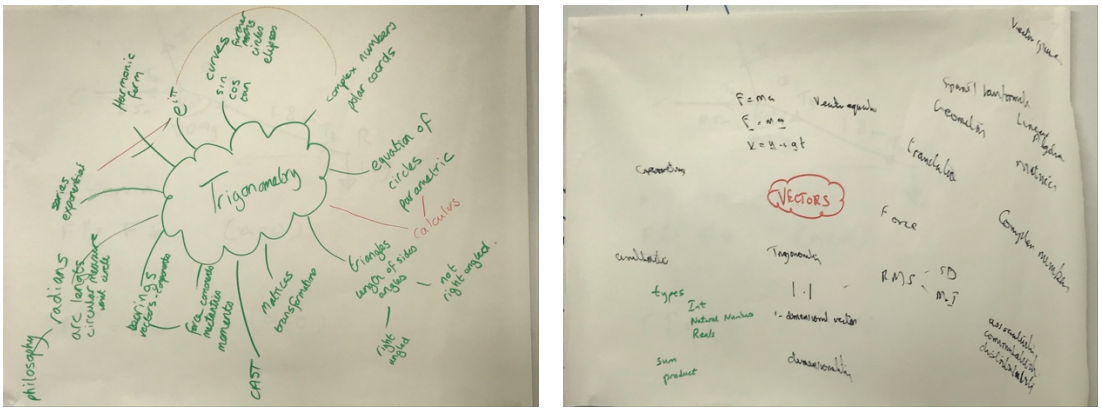



Figure 5. Trigonometry and Vectors Maps from BCME9 workshop

Paticipants comments focused on two main themes:

- how many connections there actually are
- how connections will now be emphasised explicitly.

## Conclusions

This paper has shown how we have embedded connections into our SoW, which can lead to more explicitly connected PoS and more mathematically coherent teaching.

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# English Secondary Mathematics Teachers and Digital Technology

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**Keywords: secondary mathematics; teachers; digital technologies**

## Introduction

In the early days of computers in schools (pre-1980), only some secondary schools had access to mainframes held in university, polytechnic or local council. Input was by punched cards or via a dedicated telephone line, and after a few days the results of the punched card work would be returned (Millwood, 2009). This paper follows some of the main events from that time to the present day with particular reference to constraints and training.

## Early Days

The first real curriculum access was in about 1979/1980, following the Labour government vote in 1979 to make a major investment in computer access for schools and was co-funded by the Department of Trade and Industry (DTI) and the Department for Education (DfE). The government restricted the companies from which purchases could be made, to support the developing British computer industry, to Research Machines (RM), British Broadcasting Company (BBC) and Sinclair Computers. They were expensive and schools were unable to afford many (a RM480z model retailed for over £500, BBC Model B about £300 and the Sinclair Spectrum 16KB RAM version sold in 1982 for £125 with the 48KB version at £175 (excluding disk drives and monitors (http://www.computinghistory.org.uk)), so initially one or two machines was the norm in secondary schools. Software was limited and initially aimed at business rather than education.

The training focus and priority was on technical competence, rather than pedagogy and the opportunities afforded by using technology. Initially schools were offered training places for two teachers for two days who were then meant to 'cascade' to the rest of the staff. For teachers who were not technically minded or were part of the group of early adopters and innovators (Rogers, 1983) this introduction was not inspirational and opportunities were lost to engage these teachers, some of whom are still teaching, in the early stages. Much of the early software was written by the early adopters and innovators and shared with other teachers through the introduction of the Microelectronics Education Program (MEP) (Fothergill, 1981) and the Secondary Mathematics Individualised Learning Experiment (SMILE) programme developed by Inner London Education Authority (ILEA) teachers for RM machines from 1979 and BBC B (1984) (http://www.naec.org.uk/artefacts/software/micro-smile).The advent of commercial companies such as Longman provided teachers with education specific software rather than that designed for business. Training was frequently offered by Local Education Authority Advisory teams, and accessed through schools releasing staff, or subject association meetings and magazines including The Mathematical Association (MA) and the Association of Teachers of Mathematics (ATM) (Pimm and Wilder, 2005). This limited access for many teachers.

Another opportunity was lost following the 1994 curriculum review, when certain software was prescribed impacting teacher development of software resources. Little money was made available for updating teachers' computer skills or providing

facilities to train mathematics teachers in applications, so learning to use software was largely left to teachers themselves. The relatively small number of incidences of digital technologies in the National Curriculum did not make learning to use Logo, databases and graph plotting software for the classroom a high priority for many teachers. There was little incentive for the non-enthusiasts to spend time learning how to use them.

Widescale training was offered by the government via the National Opportunities Fund (NOF), which all teachers were expected to undertake. However, lack of consultation with teachers about their needs resulted in a scheme that focussed on administration and engaging with emerging use of the internet and email rather than pedagogy with practical support for use of computers in the classroom. Coupled with the government's expectation that teachers used their own time for NOF training did not result in widespread engagement (Kirkwood et al., 2000).

## Constraints

If there is to be more engagement, a number of obstacles need to be overcome such as the effect of teacher beliefs, an increasingly prescriptive curriculum not reflecting life in the digital age, the control exerted over teachers through targets, performance and accountability, teachers becoming risk-averse in a high workload, top-down, judgmental management system and lack of effective training and support. Dealing with any one issue in isolation does not solve the problem. While teachers need the support of others to become confident and competent, other factors need to be addressed such as having good access to resources at home and school with speedy technical support and training to reduce the fear of things going wrong. Much early training was focussed on technical issues but training in skills and pedagogy needs to be provided (Jones, 2004) with support from knowledgeable 'others' and the opportunity to work with others.

## Curriculum issues

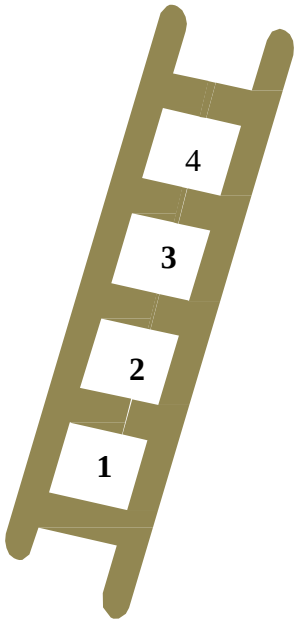
Until the National Curriculum (NC) was introduced teachers and schools were freer to create and develop their own curriculum and this enabled innovation. Within institutions there has been focus on the content and technical aspects of using ICT rather than pedagogical practice (Webb, 2002). According to ACME (2011) use of digital technologies enables abstract ideas to become manipulable and through dynamic representations effects such as geometric relations and properties become accessible. It also brings mathematics education into line with out-of-school situations and experiences.

However prescribing software to be used has narrowed the range many teachers incorporate into the curriculum. Following each revision of the NC less importance has been attached to using digital technology for teaching. The relatively small number of incidences of IT in the NC did not make learning to use Logo, databases and graph plotting software for the classroom a high priority for many teachers, so little incentive for the non-digital technology enthusiasts to spend time learning how to use them and with little money for updating teachers' computer skills or providing facilities to train mathematics teachers in applications, learning to use software was largely left to teachers themselves. Smith et al. (2008) mentioned teachers were not convinced of the benefits of using ICT with the result that many teachers were not aware of alternative, effective, methods of delivering lessons. Like Forgaz (2002), my research (Parish, 2018) found that secondary mathematics teachers use a limited range of software: generic (Word, Excel, PowerPoint), dynamic geometry (Cabri, Geometers' Sketchpad,

GeoGebra), graphing (Autograph, GeoGebra), the internet (YouTube video) and subscription sites such as MyMath and Manga High. Logo, once explicitly mentioned in the mathematics NC, is hardly used. My study also showed that these teachers rated highly: student engagement, interactivity, visualisation and the way students could receive instant feedback as reasons for using digital technology. A group of 34 first year mathematics undergraduates also rated highly: interactivity, visualisation and instant feedback.

## Training

Although there are several forms of training including workshops, cascading, self-teaching, manuals and internet the participants in my study showed a preference for learning with others. However, one of the difficulties cited was teachers having time away from the classroom for external training events. A cheaper alternative adopted in many schools is the cascade model or in-house training but the problem is summarised in figure 1 that, according to Miller et al. (2008: 13), is attributed to Dubin (1962) where there is a presupposition that the lead person is knowledgeable and competent.




**4 Unconscious Competence:** At this point using the skills becomes a habit, but there needs to be awareness of complacency, also being supportive of those still at stage 2.

**3 Conscious Competence:** At this stage it is about keeping focused on effective performance of the task, with plenty of practice so the new skills become automatic.

**2 Conscious Incompetence:** During this stage people need plenty of encouragement as they find out they do not know as much as they thought they did.

**1 Unconscious Incompetence:** These people need to realise how much there is still to learn.

Figure 1 Conscious Competence Ladder

This demonstrates the potential risk of in-house cascade training, just because someone uses a piece of software or hardware they are not necessarily able to deliver quality training. Sending a person on a one or two-day course is unlikely to have resulted in the level of expertise required to support colleagues in how to use the same ideas in their classrooms.

Recommendations for the amount of time needed to learn to use software suggest much more than teacher and school expectations. NCETM recommends six hours in split sessions while the earlier Stevenson report (1997) suggests 20 to 30 hours and with McKinsey and Co. (1997) up to 60 hours. The 22 teachers in my research thought that one or two days was adequate to learn how to use software sufficiently well to be competent in the classroom. Success will also be based on their receptiveness to the course presenter, the content, training approach, ability to follow the course and the manner in which they follow-up the course with or without support.

## How might more teachers become engaged with digital technology?

It is difficult to ask teachers to devote time to upskilling themselves and become digitally confident and competent but there are routes that can be followed. Apart from offering quality professional training, one of the strongest ways is building communities of practice not only within a school but through social media or online groups offering discussion about use in the classroom and its effectiveness. Encouragement can also be given by integrating activities into text books/teaching resources that support and challenge students' thinking beyond the capabilities of traditional questions and enabling creativity within published/prescriptive schemes of work.

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# The Missing or Unknown Polyhedra

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**Keywords: polyhedra; practical geometry; problem-solving**

## Introduction

This paper presents a few from a plethora of new polyhedra that arose from asking the question: *What If Not?* (Walter & Brown, 1969) about the established polyhedra that are predominantly made from regular polygons. They are coined as the PINEL polyhedra, from the acronym Polyhedra If Not Equi-Lateral. This began with deltahedra formed from congruent isosceles triangles (CIT) and congruent scalene triangles (CST). The polyhedral have been constructed from ATM MATs (Pinel, 2010). Some of these new polyhedra involve regular polygons too. In addition, the cut-pentagon shapes (Gailiunas and Pinel, 1996), especially the two isosceles triangles.

## Deltahedra - polyhedra made entirely with triangles

Among triangles the equilateral is a special case - a one-off - whereas isosceles triangles abound and scalene triangles even more so. Among geometrical forms 2D shapes are seen most often in the world as faces or facets of 3D objects. Strange and perverse then that the school mathematics curriculum looks predominantly at 2D, and within this spends most of the very little time on geometry considering special cases, like the equilateral, and the right angled. It seems to me that this is in part because so little is made available to us about the multitude of relatively simple 3D forms that can be readily accessed. My study - in part - is driven by the need to change this, but also by mathematical curiosity and the mathematising it energises about the possibilities that are 'just around the corner' if we ask *What if Not?* (Walter and Brown, 1969)

### *Tetrahedra: polyhedra made from just four triangles*

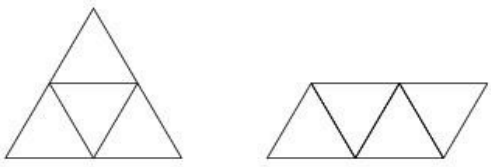


Figure 1. Nets of a regular tetrahedron

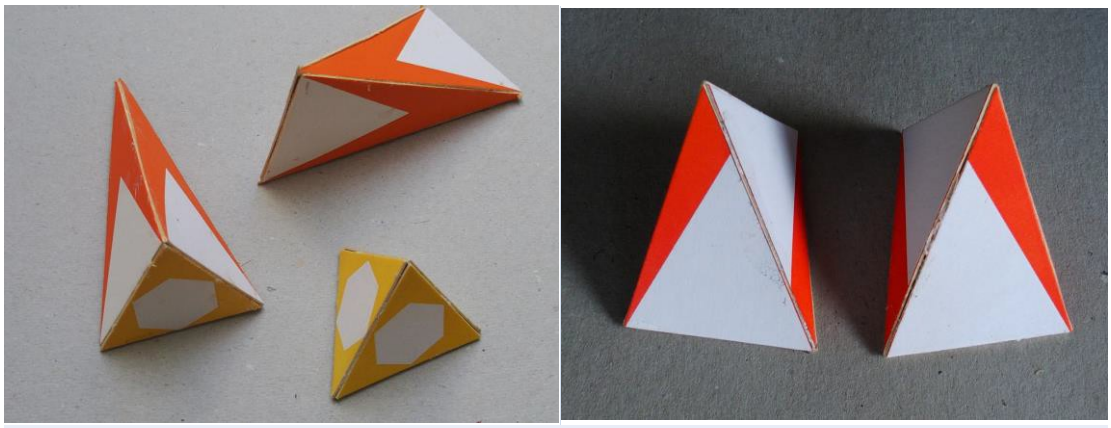
How many nets are possible with four congruent equilateral triangles (CETs)? (see figure 1)

Suppose instead you have four congruent isosceles triangles (CITs e.g. the isosceles triangular MAT), or scalene triangles (CSTs e.g. trim an isosceles triangle MAT)… *How do the nets change?*

The five possible tetrahedra are illustrated in figure 2 below. Another tetrahedron - a special case - can be made from one regular pentagon and one equilateral triangle. The pentagon forms three of the faces.

*Could any other equilateral pentagon be used with an equilateral triangle to make a tetrahedron?*

*Why are there no examples with four right-angled or four obtuse-angled triangles?*



CET and CIT Tetrahedra | The pair of CST Tetrahedra

Figure 2. Tetrahedra using Mathematical Activity Tiles (MATS)

### *Hexahedra and Octahedra*

To move from four to six triangles seems a simple step. *Why not try with five triangles?*

If you put two tetrahedra together at a matching face, instead of $2 \times 4 = 8$ faces, you hide two faces, so there are six. All the triangle based hexahedra arise from this method. *With the five tetrahedra we already have, how many pairings can you make to form hexahedra?* A few are shown in figure 3.



Figure 3. Five of the hexahedra formed by joining matching faces of two tetrahedra

Moving on to Octahedra, this is where some polyhedra appear that are not in any book nor on any web-site. The more obvious octahedra are shown in figure 4.



Figure 4. Five of the relatively obvious octahedra, although the final 'boat' octahedron is less obvious

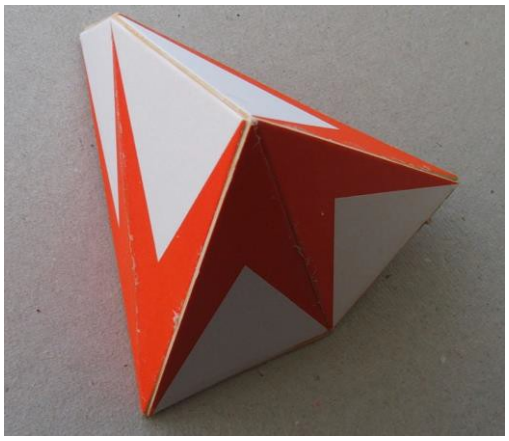


Figure 5. CIT octahedron with kite-like pairing

Starting with eight CITs, you are likely to recognise that they will make square based pyramids which when mirrored resemble a regular octahedron, but stretched. There is another from the same kit of eight CITs, by first making four pairs into kite-shapes (figure 5).

More interesting are those composed of CITs and CSTs, sometimes including a few CETs, and those made from cut-pentagons and CETs. Of these the most exciting to discover were the four different CST octahedra (figure 6).

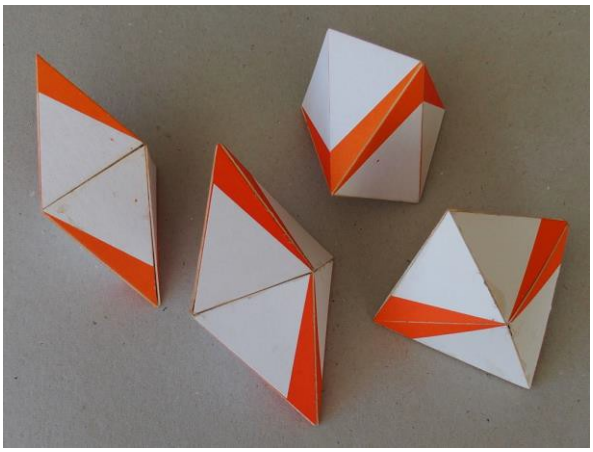


Figure 6. Four octahedra made from eight CSTs

When using eight CSTs, there is one with three planes of symmetry at right angles to each other. Then there are three others where only two planes of symmetry are evident, with the missing plane replaced by alternating up and down vertices. As there are three different sides, these arise from one of the sides being the alternator.

Next we have four all-new CST octahedra. In addition, two of these can be made with right-angled CST and obtuse-angled CST (cut from Rectangle MATs).

*Why can the other two not be made with right-angled triangles?*

Other octahedra kits worth exploring are:

- Six CSTs and two CETs (*three found*)
- Eight CITs (obtuse – from cut pentagons);
- Four CITs (obtuse – from cut pentagons), four CETs;
- Four CITs (obtuse – from cut pentagons), two pentagon trapezia, two CETs (*three found*);
- Five CETs, three regular pentagons.

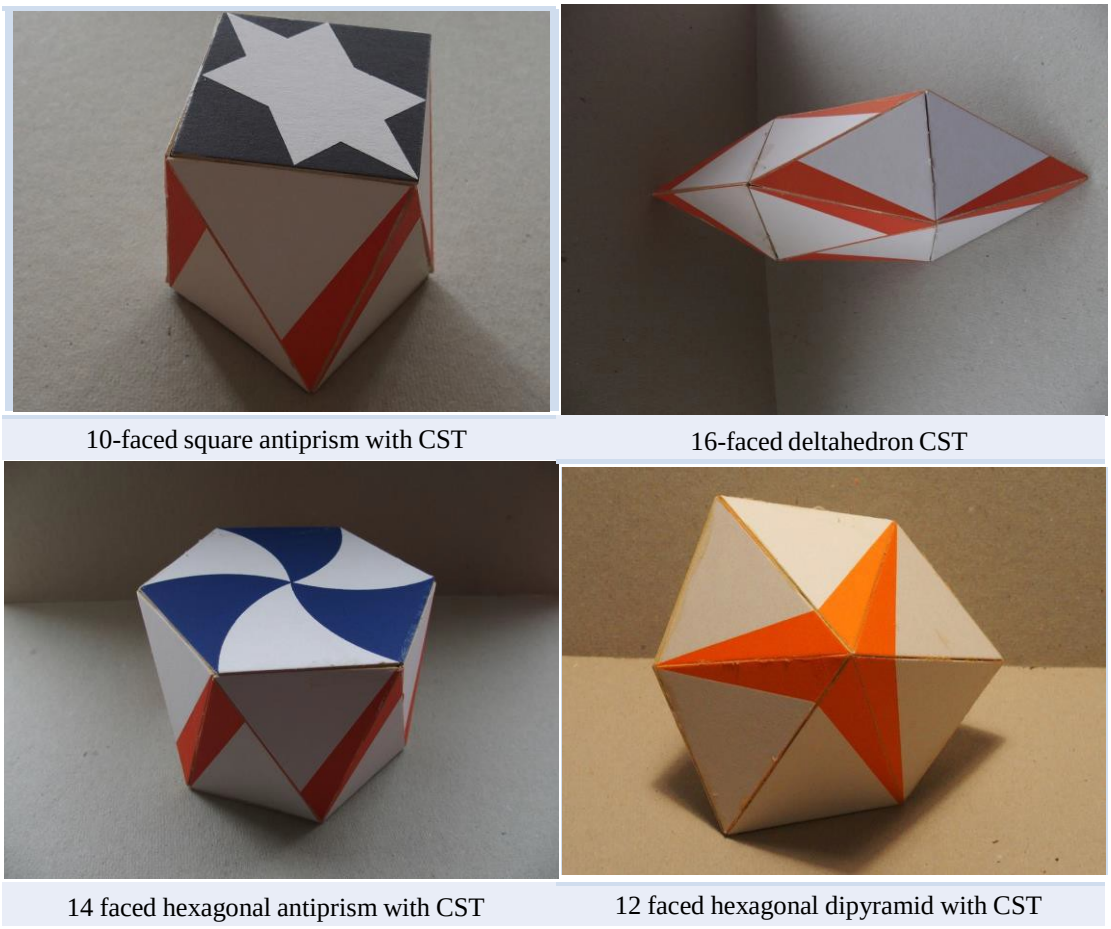
The polyhedra found so far are important building blocks for others including those with more faces.

### *New polyhedra with more than eight faces using CST*

To move beyond eight faces with scalene triangles was a step into the unknown. There are three important findings so far: being able to build some pyramid like structures where once again reflective pairings were clearly important; also the rings of rotationally connected triangles that resembled the form of antiprisms. In discussion separately with Roger Beeney and Matthieu Pinel, it became clear that these could be combined. Pyramid attempts seemed to lead to non-regular bases, but that made me think of the semi-regular equilateral polygons that David Fielker (1981-1983) had included in his series on *Breaking the Shackles of Euclid*.

Combining these thoughts, I could build dipyramids, and antiprisms (some with regular polygon ‘ends’) and so found many polyhedra with pleasing symmetries and 12, 16, 18, 24 and more faces, some are shown in figure 7.

Building with MATs and latex adhesive is a skill that opens up this field. If it is new to you, there are two recent publications and associated MATs available from ATM: *A practical introduction to Geometry using MATs, Pamphlets 1 & 2*, by Geoff Faux and Adrian Pinel



10-faced square antiprism with CST | 16-faced deltahedron CST

14 faced hexagonal antiprism with CST | 12 faced hexagonal dipyramid with CST

Figure 7. A few polyhedral with ten or more faces

## Conclusions

In hindsight, it was a very good decision to start to explore the possibilities of polyhedral made from congruent scalene triangles. Paul Gailiunas and Geoff Faux were instrumental in encouraging me. Paul referred me to George Hart (n.d.). There are still many polyhedra out there to imagine and build, but also there is a much richer body of knowledge about polyhedra with less than ten faces. One consequence for me is to take this further by cataloguing the polyhedra in an accessible way that will support teachers and others who wish to browse or become more practically involved.

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# Using the history of mathematics to inspire and motivate learning

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**Keywords: history of mathematics; motivation; learning**

## Why use the history of mathematics?

Mathematics is a truly universal language, used throughout history all around the world. Despite the white, male, Eurocentric stereotype that dominates school mathematics, there is considerable potential to share mathematics from different cultures and times to inspire and motivate learning. There are well documented arguments (Fauvel and van Maanen, 2000) for using the history of mathematics in the classroom that include raising awareness of mathematics as a cultural-human endeavour and developing positive attitudes towards mathematics. It also enriches teachers' understanding of mathematics and their pedagogic strategies.

On the other hand many have their reservations: the history of mathematics is not mathematics, it may be confusing, irrelevant and boring. There is not enough time to include history of mathematics in a crowded curriculum, there are few suitable resources, teachers lack expertise and it is not assessed so it is not valued.

In Engand, the history of mathematics could be considered an entitlement rather than an option, given the Purpose Study that applies to the entire 5-16 mathematics programme of study:

> Mathematics is a creative and highly inter-connected discipline that has been developed over centuries, providing the solution to some of history's most intriguing problems (DfE, 2013).

Exploring the history of mathematics will help to develop students' "… ability to reason mathematically, and a sense of enjoyment and curiosity about the subject" (ibid.). Williams recommended that the primary curriculum should include

> opportunities for children to engage with the cultural and historical story of both science and mathematics could have potential for building their interest and positive attitudes to mathematics (2008: 255).

The current primary mathematics programme of study includes for Year 4

> Read Roman numerals to 100 (I to C) [to 1000 (M) and recognise years written in Roman numerals (Y5)] and know that over time, the numeral system changed to include the concept of zero and place value (DfE, 2013).

Which offers considerable scope for primary teachers, unfortunately there is nothing similar in the secondary (KS3 and 4) programmes of study. In this short article, I share some favourite historical images for the classroom.

## Where does mathematics come from?

Much school mathematics is old, with roots in Africa, China, India, and the Middle East. The silk route ensured exchange of ideas, culture and commerce. Mathematics is an essential part of commerce. Recording quantity was an important part of managing society. Around 5000 years ago (3000 BCE) the Egyptians and Babylonians were managing complex societies and using mathematics to help them do so.

### *Egyptian mathematics*

Learners find Egyptian numerals intuitive and easy to use – there is no place value, a new symbol is used for each power of ten and the symbols are simply repeated for quantities greater than one (figure 1).

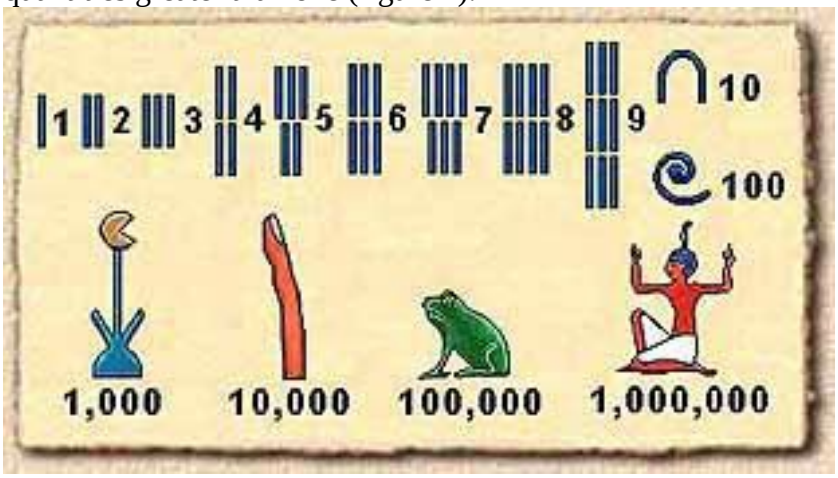
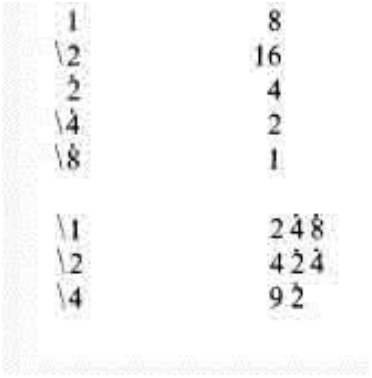



Figure 1. Egyptian numerals (pintrest.co.uk)

Egyptian arithmetic relied on being able to add/subtract, double and halve and work with 'unit fractions'. We know about the mathematics education of Egypt through relics like the Rhind papyrus, which is believed to be a 'textbook' for scribes.




There are a wealth of problems with solutions. For example Problem 24:

A quantity, $\frac{1}{7}$ of it added to it, becomes it: 19

The solution involves starting with 7, as $\frac{1}{7}$ is easy to determine. Then how many 8s make 19? Figure 2 includes a translation of the Egyptian method, $8\left(2+\frac{1}{4}+\frac{1}{8}\right)=19$ so the solution is $7\left(2+\frac{1}{4}+\frac{1}{8}\right)=16+\frac{1}{2}+\frac{1}{8}$

Figure 2 Extract from Rhind papyrus (Fauvel & Gray, 1987: 14)

This method for solving equations was used well into the Middle Ages, when it was known as the method of 'false position'.

### *Babylonian Mathematics*

About the same time as the ancient Egyptians, the peoples living in the fertile region between the Tigris and Euphrates rivers, the Babylonians had a very different number system. We know a great deal about the Babylonians as their recording was on clay tablets. They also educated scribes, and we have access to some of their work. Two personal favourites are in figures 3 and 4. They are a great starting point for playing mathematical detective – students are quick to appreciate that they involve numbers, but how did Babylonian numbers work? Like the Egyptians, they used a symbol for one and a different symbol for ten, but when they got to 59 (in base 10), they wrote 1 in a new position – they used place value - with base 60. Place value is remarkably more sophisticated than ancient Egyptian numbers, or Roman numerals. Translating the

Babylonian numbers into the Hindu-Arabic base 10 number system we use today,

requires understanding of powers and place value, as well as efficient use of a calculator when working on the Yale tablet.

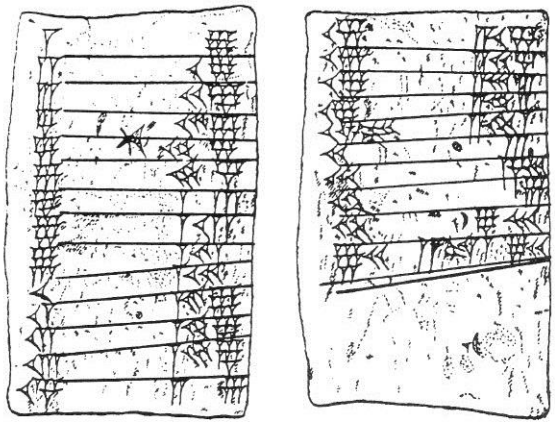


Figure 3. A table of multiples (public.csusm.edu)

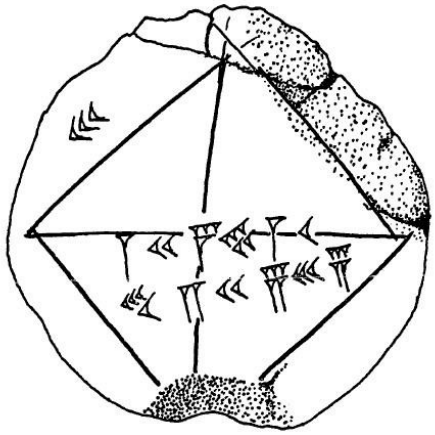


Figure 4. The Yale tablet (maa.org)

When I taught in school, we used the table of multiples in year 7, and this allowed us to work on place value, powers and the multiples of nine. In year 10, we used the Yale tablet as an alternative route into 'Pythagoras' theorem', irrational numbers and solving quadratic equations. How did the Babylonians know $\sqrt{2}$ correct to 5 decimal places? Leo Rogers has prepared a diagram summarising many of the possible links for the Yale tablet (figure 5).

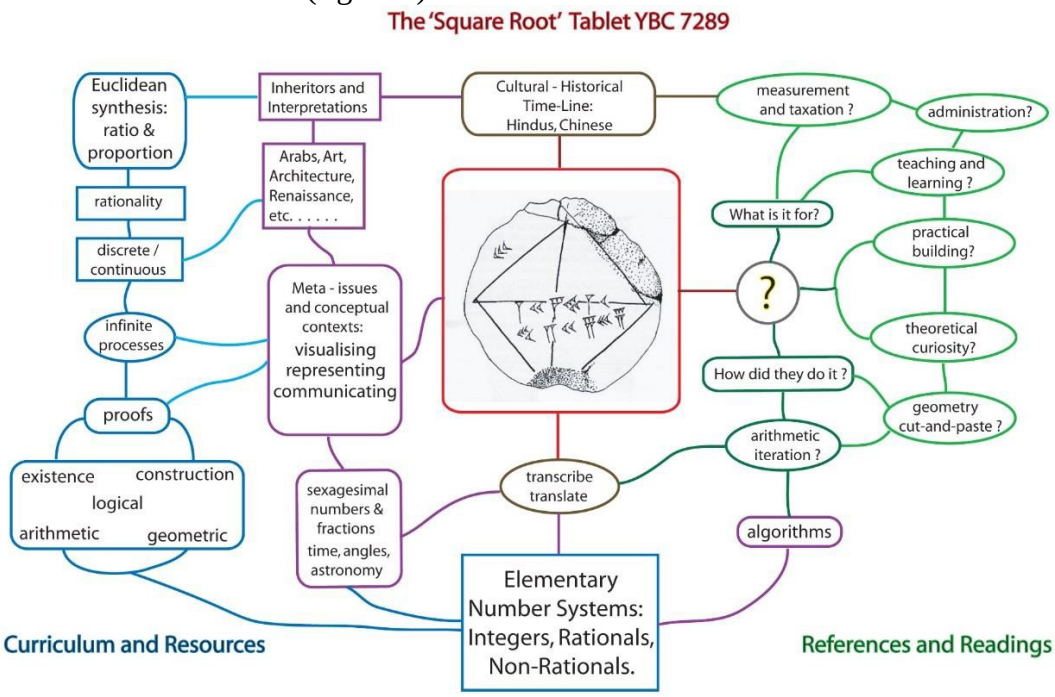
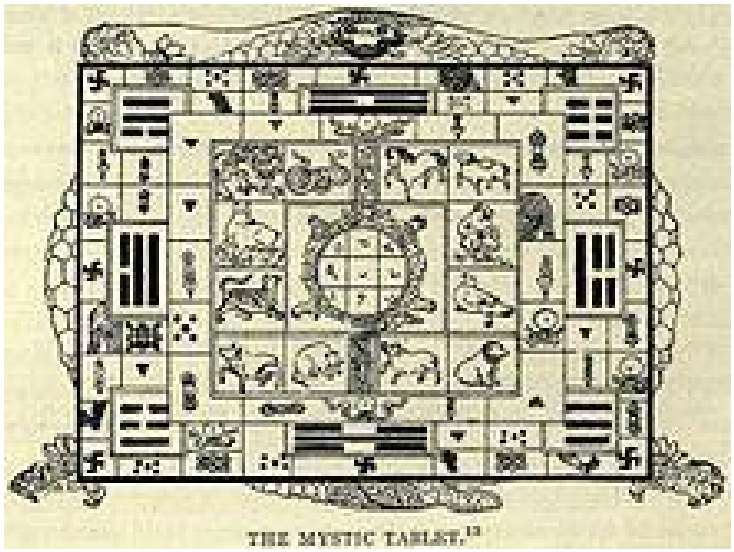



Figure 5. The rich potential of the Yale tablet

## Mathematics from China

Have you ever wondered where Magic Squares come from? Puzzle no longer, legend has it that the three by three (Lo Shu) magic square (figure 6) came on the back of a turtle from the Yangtzee river.



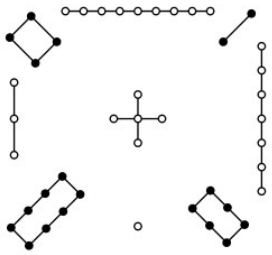



Figure 6. Magic squares come from China (pintrest.com)

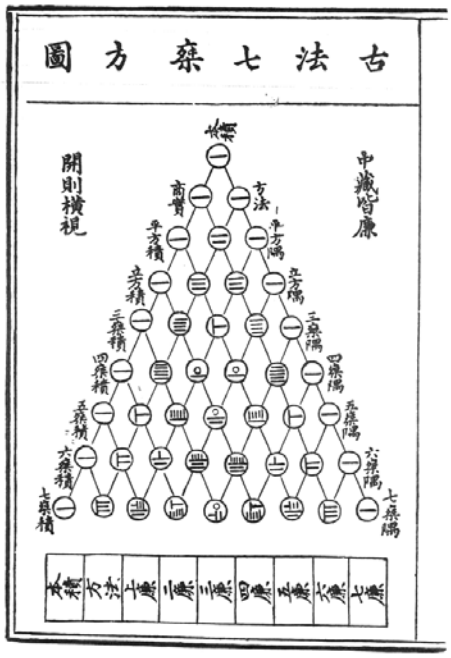



Figure 7. Yang Hui's triangle, 1303 (ipsf.io)

A popular misnomer is 'Pascal's triangle'. Pascal published a book about the arithmetic triangle in 1653 but it was well known in China, India and the Middle East hundreds of years before. A Chinese version is another great starting point for learners to be mathematical detectives (figure 7). The system for recording numbers is different again – how does it work?

There are many ways in which the numbers in the triangle (the binomial coefficients) are used in mathematics, from the coefficients of expanding $(a+b)^n$, or determining the number of ways of selecting $r$ items from $n$ $\binom{n}{r}$.

There are numerous patterns in this triangle that learners can explore.

## Conclusion

I hope this short article inspires you to explore the history of mathematics.

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# An Appeal to the Visual in Algebra

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**Keywords: algebra; factorisation; visual representation**

## Introduction

The advantages of multiple-representation in the learning of mathematics have long been promoted. Not only has the obvious need for the sensory-impaired student to access mathematical information through as many different channels as possible been extended to all students but having a variety of visual inputs is now seen as a way to optimise possibilities for understanding. For a student who finds one approach inaccessible, another approach may just open the door. And all students will have the opportunity to make connections that might not be at all apparent if a single representation is adopted by the teacher. Here we consider ways to visualise just three standard results – a difference of squares, a difference of cubes and a sum of cubes.

## Difference of squares

The difference of squares formula can be derived using areas in at least three ways, the easiest of which is shown in figure 1. We remove a pale yellow square of side $y$ from a corner of a square of side $x$. Now slice a rectangle from the upper part of the figure as shown and reattach it on the right-hand side. This gives a rectangle of sides $x + y$ and

$x - y$. So $x^2 - y^2 = (x - y)y + (x - y)y = (x - y)(x + y)$.

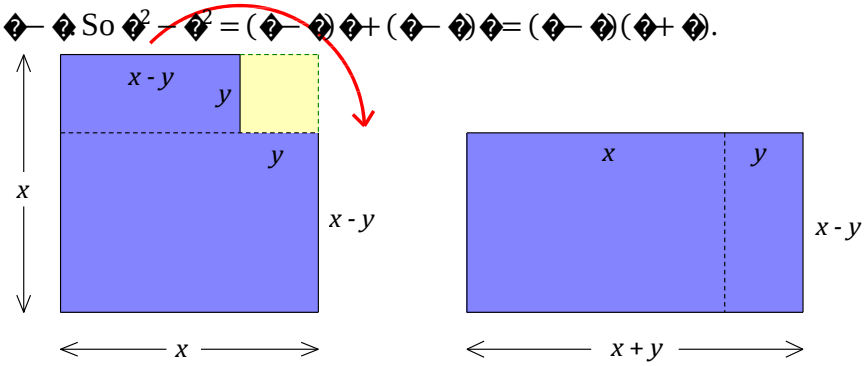
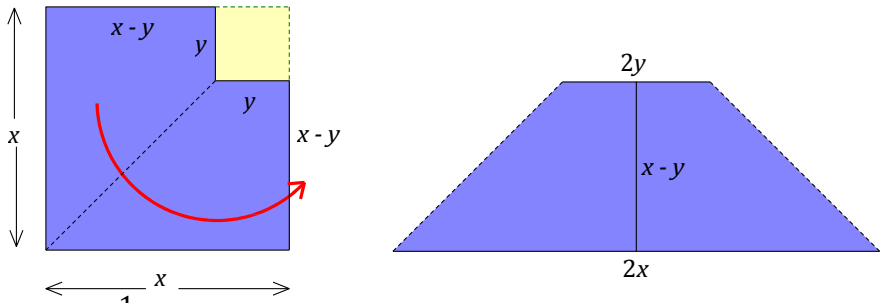



Figure 1. First rearrangement of a difference of squares

Notice that part of the original figure has height $x$ and the other part has height $x - y$ where the final rectangular figure has a uniform height, $x - y$ which is the common factor in the second step of the associated algebra.

If we make the cut in the original figure in a different place we can rearrange the pieces into an isosceles trapezium (figure 2).




$$x^2 - y^2 = \frac{1}{2}(2x + 2y)(x - y) = (x + y)(x - y).$$

Figure 2. Second rearrangement of a difference of squares

Of course, rearranging into a rectangle is even simpler, as figure 3 depicts.

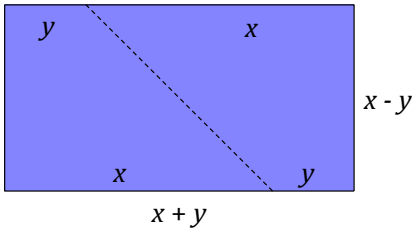



Figure 3. Third rearrangement of a difference of squares

## Difference of cubes

Now we remove a cube of side *y* from a cube of side *x*. This could leave a shape such as the one shown below, with its 'missing corner'. We slice as indicated by the dotted lines in figure 4 to produce three cuboids.

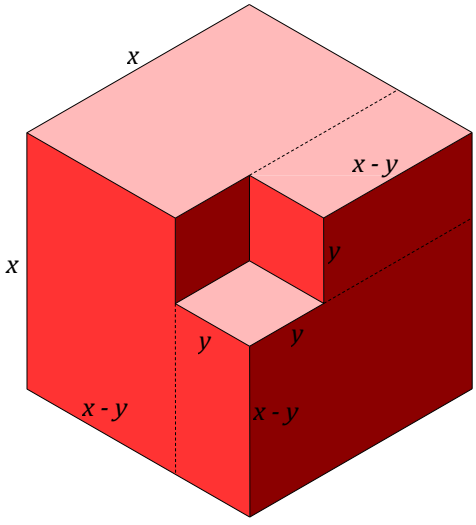
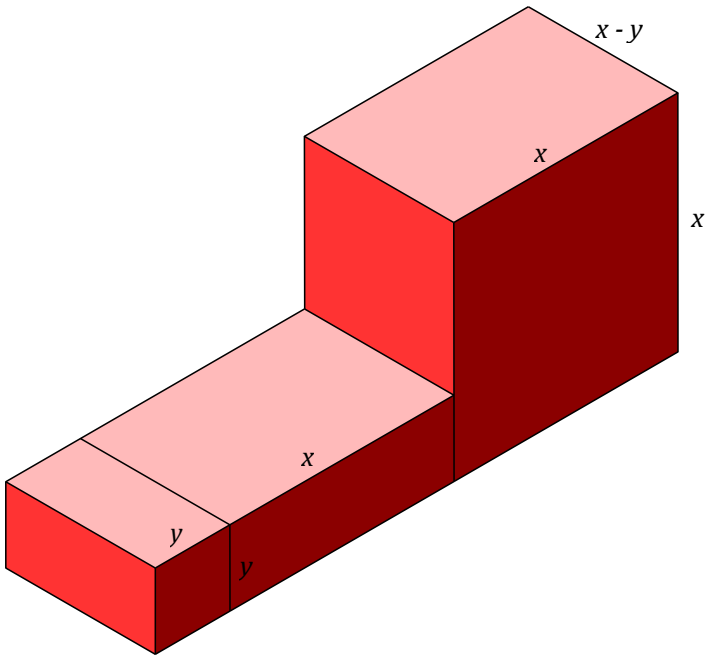



Figure 4: Dissecting a difference of cubes

The volumes of the three cuboids, in descending order of magnitude, are $(x - y)y^2$, $(x - y)xy$ and $(x - y)^2y$. They have one dimension, $x - y$, in common and this is both the common factor of the factorised expression for the difference of cubes

but also the length of the L-shaped prism created by bonding the blocks together in a new configuration (figure 5).




$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Figure 5: First rearrangement of a difference of cubes

Of course, there are other ways in which a difference of cubes could be sliced to produce the factorised form of the volume. We will restrict ourselves to just one of them, beginning with the smaller cube (outlined in blue) as a hole symmetrically positioned in the middle of the larger cube (as shown on the left of figure 6).

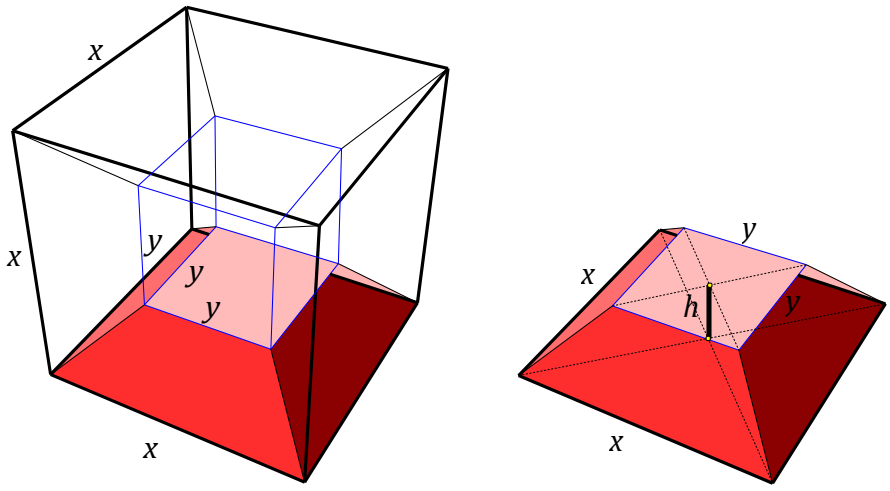



Figure 6: From a difference of cubes to the volume of a frustum

On the right, a truncated square-based pyramid or *frustum* has been isolated. Let us find the volume of that frustum by slicing it into pieces with four downward cuts parallel to the edges of the base. This creates nine pieces – a cuboidal central block, four side pieces in the shape of a triangular prism, which can be recombined to make a cuboid, and four corner pieces which make a square-based pyramid (figure 7).

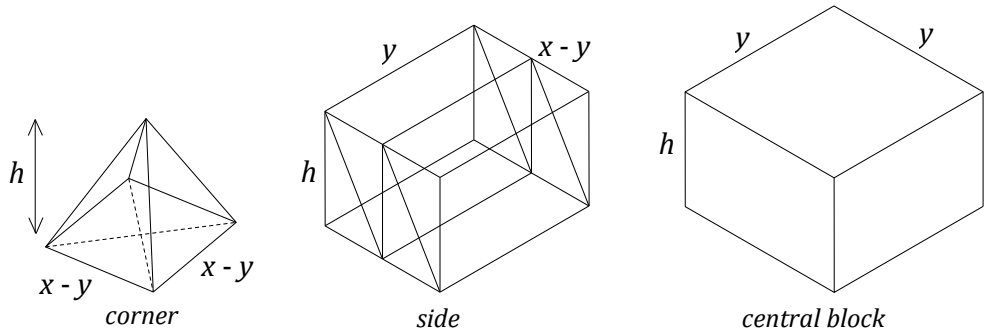



Figure 7: Dissecting and rearranging a frustum

The volume of the frustum is $\frac{1}{3}(x-y)^2 h + (x-y)yh + y^2 h$ which simplifies to $\frac{1}{3}h(x^2 + xy + y^2)$. But $h = \frac{1}{2}(x-y)$, so this volume is $\frac{1}{6}(x-y)(x^2 + xy + y^2)$. In the whole figure, there are six such frustums, so $V = (x-y)(x^2 + xy + y^2)$. As we now know this volume two ways, we again have $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$.

**Sum of two cubes**

Finally, we often forget that $x^3 + y^3$ can be factorised as $(x+y)(x^2 - xy + y^2)$. One way to provide a visual proof is by using figure 8 below. This time, it is more difficult to see how the pieces should be repositioned so that the linear factor reveals the quadratic. Hint: start by making two cuboids, one of them using pieces 1, 2 and 3.

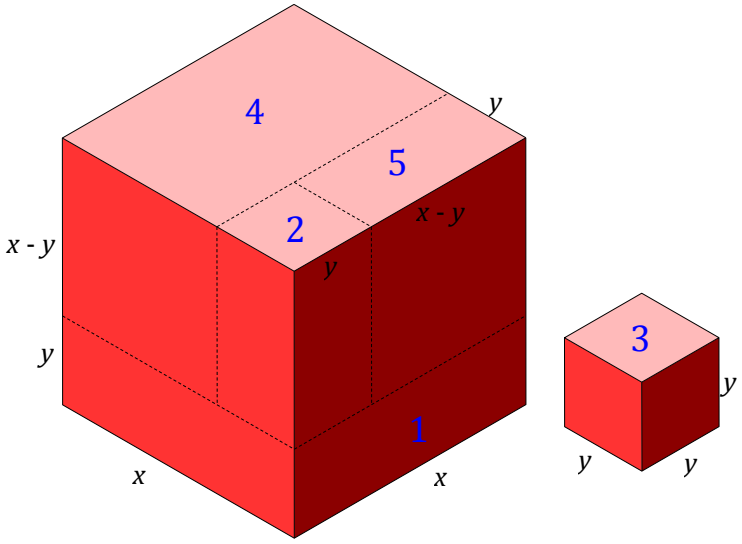



Figure 8: Cutting and rearranging to factorise a sum of cubes

These are just a few ways in which algebraic results can be visualised. More of them will appear in the author's book, *The Room in the Elephant*, which is in preparation. Meanwhile, it is hoped that teachers will explore some possibilities themselves as they prepare to support their students to grasp key relationships.

# Some Calendar Problems

Anthony C. Robin

*Independent*

**Keywords: calendar; day of the week; date of full moon; date of Easter**

## Finding the Day of the Week

We often need to know the day of a certain date, or vice versa. There have been various rules devised for doing this, we shall look at two:

### *Mental Method*

Devised by mathematician J H Conway devised this. If we know the weekday of any day in a month, adding or subtracting seven to that date will have the same weekday, and other dates can just be found by counting on or back.

Next, for any year, Doomsday is the day of the last day of February. For 2018 it is Wednesday, and for 2017 it is was Tuesday, so for 2019 it will be Thursday, and 2020 it will be Saturday. Once the Doomsday for a (non-leap) year is known, we can find all the other dates as follows: 31 January is on a Doomsday. For other even months, the Doomsday is on the nth day of the month i.e. 4 April, 6 June, 10 October and 12 December. For other odd months it is either the n+4th day for the long months, and on the n-4th day for the short months, September and November. This means that Doomsday is 7 (3+4) March, 9 May, 11 July, 5 (9-4) September and 7 November. In a leap year, all dates must be shifted forward one day.

Once the Doomsday for a year is known, the other dates for the year can be found fairly easily. Each year the Doomsday moves on one day except for leap years where it moves on two days. For the year 2000 it was a Tuesday, and for 1900 it was a Wednesday. To find the Doomsday in another year, we need to know the number of leap years, this is found by dividing by 4 (ignoring any remainder). For example, to find Doomsday in 2035: 35 divided by 4 is 8, remainder 3, the doomsday for 2035, will have moved on from 2000 a total of 35 + 8 = 43 days. As 43 divided by 7 is 6 remainder 1, this is equivalent to one day, so its Doomsday will be Wednesday.

The Gregorian Calendar, which is used throughout most of the World, has leap years when the year is a multiple of four, unless it is a multiple of 100. In that case it is only a leap year if it is also a multiple of 400. So 1996 was a leap year as was 2000, but 1900 was not. This calendar was devised in 1582 by Pope Gregory, but was only introduced into Britain (and American colonies in 1752). The earlier Julian calendar had the same structure for each year, but a leap year occurred every fourth year.

This means that the Doomsday for 2100, will have moved on from 2000 a total of 100 + 24 = 124 days, which is equivalent to 5 days, or two back so that the Doomsday for 2100 will be Sunday. For 2200, another two days back so will be Friday, for 2300 another two days back so a Wednesday. But then because 2400 will be a leap year, it will have a doomsday of Tuesday, so that the whole calendar repeats every 400 years.

### *Zeller's algorithm*

This is more convenient if we are wishing to program a rule into a computer, or calculator. For some purposes the year was considered to start not in January but in March. We have two remnants of this in our modern calendar, firstly that the leap year

day is added at the end of February which would have been at the end of the year, and secondly the names of the months at the end of the year, September, October, etc. have names from the numbers 7 (sept), 8 (oct), but in our calendar they are the 9th and 10th months. For this algorithm January and February are treated as the 13th and 14th month of the previous year. We then calculate the value of the formula:

(D) + [2.6 × (1+M)] + Y + [Y ÷ 4] + [Y ÷ 400] - [Y ÷ 100] Modulo 7

Where D stands for the day of the month, M the month, and Y the year; and [*x*] means, the integer value of *x*, rounded down, e.g. [5.3]=5, [7.9]=7, [6]=6, [-4.2]=-5. Modulo 7 means that we take the remainder after dividing by 7. 1 means Sunday, 2 Monday, etc. with 0 or 7 meaning Saturday. This formula can easily be adapted for a spreadsheet using the INT (equivalent to [*x*] and MOD (gives the remainder on division e.g. MOD(12,5)=2, MOD(21,3)=0) functions.

In 2009, the ATM conference started on 6 April – what day was that?
6+[2.6×5]+2009+[2009÷4]+[2009÷400]-[2009÷100]=6+13+2009+502+5-20 = 2515
This is then divided by seven to give 359 and a remainder of 2, so it was Monday.

*Counting the Days*

It is sometimes useful to know how many days pass between two successive events. One way of doing this is to modify Zeller's algorithm and to calculate the value of the expression:

D + [30.6 × (1+M)] + 365 × Y + [Y ÷ 4] + [Y ÷ 400] - [Y ÷ 100]

**Moon Calendar**

Living at a fairly high latitude, $51\frac{1}{2}°$N for London, the seasons have a major effect on our lives. For people living near the Equator, there is less difference between Summer and Winter, and so it is not surprising that some people in these regions have lunar calendars. From our point of view, the Moon's motion has two effects, firstly Easter and dates associated with it, and secondly the tides, which are of fundamental importance to sailors.

The time of the moon's motion around the Earth varies slightly between successive full moons. However the average period is known accurately to be 29 days, 12 hours, 44 minutes, 2.9 seconds or 29.53058912 days [1]. This figure does not divide easily into the number of days in a year, but 19 years are very close to 235 lunar months. This is called the Metonic Cycle after the Athenian astronomer about 432 BCE.

Because of the Metonic cycle, of approximately 19 years, a number called the Golden Number (GoldN) is defined. It takes the values from 1 to 19 and it is found by adding one to the remainder when the year is divided by 19. Once the Golden Number is known the dates of the Full Moons for the year are given approximately in table 1 (it is possible to be up to two days out from these dates). You may notice that sometimes a calendar month contains two full moons, the second is sometimes called a Blue Moon, and occasionally February has no full moon at all.

| Year | GoldN | Jan | Feb | March | April | May | June | July | Aug | Sept | Oct | Nov | Dec |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 1900 | 1 | 15 | 14 | 15 | 14 | 13 | 12 | 11 | 10 | 8 | 8 | 6 | 6 |
| 1901 | 2 | 4 | 3 | 4 | 3 | 2 | 1 | 1,30 | 29 | 27 | 27 | 25 | 25 |
| 1902 | 3 | 23 | 22 | 23 | 22 | 21 | 20 | 19 | 18 | 16 | 16 | 14 | 14 |
| 1903 | 4 | 12 | 11 | 12 | 11 | 10 | 9 | 8 | 7 | 5 | 5 | 3 | 3 |
| 1904 | 5 | 31 | - | 2,31 | 30 | 29 | 28 | 27 | 26 | 24 | 24 | 22 | 22 |
| 1905 | 6 | 19 | 18 | 19 | 18 | 17 | 16 | 15 | 14 | 12 | 12 | 10 | 10 |
| 1906 | 7 | 9 | 8 | 9 | 8 | 7 | 6 | 5 | 4 | 2 | 2 | 0 | 0 |
| 1907 | 8 | 28 | 27 | 28 | 27 | 26 | 25 | 24 | 23 | 21 | 21 | 19 | 19 |
| 1908 | 9 | 17 | 16 | 17 | 16 | 15 | 14 | 13 | 12 | 10 | 10 | 8 | 8 |
| 1909 | 10 | 6 | 5 | 6 | 5 | 4 | 3 | 2 | 1,31 | 29 | 29 | 27 | 27 |
| 1910 | 11 | 25 | 24 | 25 | 24 | 23 | 22 | 21 | 20 | 18 | 18 | 16 | 16 |
| 1911 | 12 | 14 | 13 | 14 | 13 | 12 | 11 | 10 | 9 | 7 | 7 | 5 | 5 |
| 1912 | 13 | 3 | 2 | 3 | 2 | 1,31 | 30 | 29 | 28 | 26 | 26 | 24 | 24 |
| 1913 | 14 | 22 | 21 | 22 | 21 | 20 | 19 | 18 | 17 | 15 | 15 | 13 | 13 |
| 1914 | 15 | 11 | 10 | 11 | 10 | 9 | 8 | 7 | 6 | 4 | 4 | 2 | 2 |
| 1915 | 16 | 30 | 29 | 30 | 29 | 28 | 27 | 26 | 25 | 23 | 23 | 21 | 21 |
| 1916 | 17 | 18 | 17 | 18 | 17 | 16 | 15 | 14 | 13 | 11 | 11 | 9 | 9 |
| 1917 | 18 | 8 | 7 | 8 | 7 | 6 | 5 | 4 | 3 | 1 | 1,31 | 29 | 29 |
| 1918 | 19 | 27 | 26 | 27 | 26 | 25 | 24 | 23 | 22 | 20 | 20 | 18 | 18 |

Table 1: Dates of the full moon over the Metonic cycle (to within two days)

## Easter

Easter was defined in 325 at the Council of Nicaea, as the first Sunday after the first full moon on or after 21 March (the Spring Equinox). This means that the earliest possible date is 22 March (1818 and 2285) and the latest is 25 April (1943 and 2038) with full moons 20 March and Monday 19 April respectively).

To find the date of Easter there are three possibilities:

1. Use a table listing all years in the period required.
2. Use an algorithm, which needs a small table of parameters valid for different eras
3. Use an algorithm that always works.

### *Algorithms*

Gauss developed an algorithm, which needs a small table of values for different centuries. Between 1900 and 2099 it is equivalent to the following:

a=MOD(Y,19) b=MOD(Y,4) c=MOD(Y,7)

d=MOD(19a+24,30) e=MOD(2b+4c+6d+5,7)

If d+e < 10, then Easter is (d+e+22) March, otherwise the (d+e-9) April.

There are a couple of exceptions to this formula, If the date given is 26 April, Easter is 19 April. If the date given is 25 April with d=28, e=6 and a>10 then Easter is the 18 April. For example the date of Easter in 2007: a=12, b=3, c=5, d=MOD(228+24,30)=12, e=MOD(6+20+72+5,7)=5, so 8 (12+5-9) April.

Meeus quotes an algorithm from Butcher (1876) that is claimed to work for all Gregorian dates, it is:

a=MOD(Y,19) b=INT(Y÷100) c=MOD(Y,100) d=INT(b÷4)

e=MOD(b,4) f=INT((b+8)÷25) g=INT((b-f+1)÷3)

h=MOD((19a+b-d-g+15),30) i=INT(c÷4) k=MOD(c,4)

L=MOD((32+2e+2i-h-k),7) m=INT((a+11h+22L)÷451)
Finally month=INT((h+L-7m+114)÷31) date=MOD((h+L-7m+114),31)+1
Applying this to 2008:
a=13 b=20 c=8 d=5 e=0 f=1 g=INT(20÷3)=6
h=MOD((247+20-5-6+15),30)=MOD(271,30)=1 i=2 k=0,
L=MOD((32+0+4-1-0),7)=MOD(35,7)=0 m=INT((13+11+0)÷451)=0
Month=INT((1+0-0+114)÷31)=INT(115÷31)=3
date=MOD((1+0-0+114),31)+1=MOD(115,31)+1=23. So 23 March.

### *Method using Full Moon table*

Alternatively we could use table 1, having found the Golden Numbers for 2007 and 2008 -13 and 14 respectively. The dates for the first full moons on or after the Equinox are, from the table, 2 April and 22 March. We know 4 April 2007 is Wednesday, so the next Sunday will be another four days on 8 April. For 2008 Doomsday is Friday, so 7 March is Friday, as is 21 March, so Easter is 23 March, confirming the previous result. However we have already said that the full moons can vary slightly from the dates above, so if the full moon for 2008 was 20 March, then this date for Easter would have been wrong.

As we are only interested in the first full moon after the Equinox, table 2 shows these dates for each Golden number. This will be valid for all this century but dates vary slightly subsequently.

| Full moon | March 22 | 23 | 25 | 27 | 28 | 30 | 31 | April 2 | 3 | 5 | 7 | 8 | 10 | 11 | 13 | 14 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Golden Number | 14 | 3 | 11 | 19 | 8 | 16 | 5 | 13 | 2 | 10 | 18 | 7 | 15 | 4 | 12 | 1 | 9 | 17 | 6 |

Table 2: dates of Easter over the Metonic cycle for the 21st century

As twelve full moons is approximately $12\times29\frac{1}{2}=354$ days, which is 11 days short of a year, we see that in the above table each year the full moon date is 11 days earlier, e.g. 3 April 3 is 11 days before 14 April 1 for Golden Numbers 1 and 2.

More accurately twelve full moons is approximately $354\frac{1}{3}$ days that is $10\frac{2}{3}$ days short of a year. So in three years the full moon will be $3\times10\frac{2}{3}=32$ days earlier (33 if a leap year is included). As another full moon is approximately 30 days later, in each three years the full moon is approximately three days earlier.

## Further Reading

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# Promoting Mathematics Literacy in Europe

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**Keywords: mathematics literacy; mathematics for the citizen; science museums; mathematics day**

## Introduction

Mathematics is a fundamental tool in our times for any professional or citizen. Nevertheless, the outcomes of European countries in international comparative studies (such as PISA and TIMSS) suggest a substantial part of European young people lack the proper mathematical skills/competencies that normally fit under *Mathematics Literacy*.

## Mathematics Literacy

The definition of Mathematical Literacy we find in PISA is:

> Mathematical literacy is defined as students' capacity to formulate, employ and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals in recognising the role that mathematics plays in the world and to make the well-founded judgements and decisions needed by **constructive, engaged and reflective citizens**. (OECD, 2016b: 28)

What does it mean for a student not be able to achieve this 'capacity'? The lowest level in PISA, Level 1, is defined as

> At Level 1, students can answer questions involving familiar contexts where all relevant information is present and the questions are clearly defined. They are able to identify information and to carry out routine procedures according to direct instructions in explicit situations. They can perform actions that are almost always obvious and follow immediately from the given stimuli. OECD (2016b: 191)

This is a very basic level and clearly is not enough for a reflective citizen. There are many students all over the world at this basic level or below. The PISA report for 2015 concludes that

> On average across OECD countries, 23.4% of students are proficient only at or below Level 1. (OECD, 2016b: 194)

Something needs to be done for this kind of students.

## The low attainers

Being mathematical literate is frequently used as a concrete 'proof" of the efficacy of the official school curriculum, as it is so important for the life of all students. OECD uses this idea to justify the investment in the PISA Studies, as we can conclude from the preface to the *PISA 2015 Results in Focus*, where Angel Gurría, OECD Secretary-General writes,

> The OECD Programme for International Student Assessment, PISA, has become the world's premier yardstick for evaluating the quality, equity and efficiency of school systems. (OECD, 2016a: 2)

I want to stress here that mathematics (and science) is important to ALL students, to ALL youngsters, to prepare them for a full citizen's life (*constructive, engaged and reflective citizens*):

> An understanding of science, and of science-based technology, is necessary not only for those whose careers depend on it directly, but also for **any citizen who wishes to make informed decisions** related to the many controversial issues under debate today. From maintaining a healthy diet, to managing waste in big cities, to weighing the costs and benefits of genetically modified crops or mitigating the catastrophic consequences of global warming, science is ubiquitous in our lives. (OECD, 2016b: 17)

When we speak of ALL students, we need to consider also the ones that are at the bottom of the PISA scale, and there are many of them:

> It is also worrying to see how many young people fail to reach even the most essential learning outcomes. (…) Only in Canada, Estonia, Finland, Hong Kong (China), Japan, Macao (China) and Singapore do **at least four out of five** 15-year-old students master the baseline level of proficiency in science, reading and mathematics. (OECD, 2016b: 4)

How can we make sure ALL students are above the baseline level in mathematics and science?

## The (lack of) Motivation

As Sung Je Cho points out, in South Korea, one of the top performing countries in PISA and TIMSS,

> Mathematics ranks the most hated subject for school children during their school years. In another words, mathematics is the most unpopular subject. People tend to think that mathematics is only needed for the university entrance examination which is a steep competition. (Cho, 2009: 1)

The reality is that students drop mathematics as soon as they can. Sung Je Cho concludes that

> (…) people usually do not realize the importance of mathematics in our daily life [and so] We have to arouse public interest toward mathematics. (Cho, 2009: 8)

Motivation to study Mathematics and Science depends on a number of factors, but low motivation gives no incentive at all:

> The main policy implication of these findings is that, in order to foster positive dispositions towards science and promote greater socio-economic diversity among students who go on to pursue scientific careers, school systems need to focus on the psychological and affective factors associated with science performance. Specific programmes might be needed to spark interest in science among students who may not receive such stimulation from their family, and to support students' decision to pursue further studies in science. (OECD, 2016b: 273)

Which are these "Specific programmes [that] might be needed to spark interest in science" so that students are motivated to study more mathematics and science? It is clear then that we need to arouse public interest for mathematics and science and its presence in our lives.

## Some proven activities

As is pointed out in a recent UNESCO report, teachers cannot do everything in the classroom

> (…) It is therefore important to afford opportunities for everyone within the school to engage freely in activities that are more open to another temporality and a different sort of didactic management. It is also important to draw on the many non-school learning opportunities open to young people today. (UNESCO, 2012: 31)

One of the most innovative activities, that already attracted more than 1 million visitors in 30 countries, is the'open mathematics' museum called ***Imaginary***, that was started as a project of the *Institute of Oberwolfach* in Germany. It intends to arouse 'curiosity' about mathematics and already includes more than 125 exhibitions, namely the *Mathematics of Planet Earth* exhibition.

> The platform www.imaginary.org sets out to evoke interest in and curiosity about mathematics. It represents a new way of communicating science to the general public: collaborative, global, free, and close to research. All material is made available to museums, schools, and a broad audience under an open source license and can thus be reproduced and used for individual/class room use, exhibitions, and events. (IMAGINARY, w/d)

There have been in the past lots of interesting initiatives like *WORLD MATHEMATICAL YEAR 2000* and *The International Year of Statistics 2013*, but they were one-time events. Other initiatives produce very useful websites, like the NRICH website (nrich.maths.org) or the +PLUS (plus.maths.org/content)/ magazine.

Other initiatives deserve praise for their originality and impact. I will just quote the MATH.en.JEANS activity (www.mathenjeans.fr) that gives opportunity to hundreds of students every year to research and present elementary mathematics new to them.

Nowadays, lots of different and original mathematics videos are available on YouTUBE that have a real impact. We can quote the series *Isto é Matemática* (*This is Mathematics*) shown on cable television in Portugal that recently won the Best TV General Programme award in the Sci-Doc festival (The European Science TV and New Media Awards 2018). These videos present a lot of different mathematics in a very funny, informal and non-technical way.

## Some new tools

All the video tools available today present new opportunities. Some are easily accessible, like the *Creator Studio* of YouTUBE that needs no installation because it is available online and allows anyone to create and distribute videos.

Using these tools, I organised a new kind of task with my students (future mathematics teachers) that explored a very recent topic from real life, the *WannaCry* virus, a computer virus that affected more than 100 000 computers in 150 countries all over the world and was protected with a cryptographic procedure. The students produced a video themselves and had to research Computer Virus, Cryptography and Ethics, to make a presentation in the classroom and, after editing the video, on the internet (youtube/7o79eQILxSI)

This way they will be better prepared to deal with their students, including the ones that are poorly motivated, because they are able to produce new materials with digital tools that offer new ways of connecting with their students.

## Teacher preparation

The need to popularise mathematics and motivate young learners is not completely new (remember the popular *Mathematics for the million* by Lancelot Hogben of which Albert Einstein said, "It makes alive the contents of the elements of mathematics.") but

it has new aspects today, when we are constantly bombarded with numerical, graphical and statistics information that frequently is badly described or wrongly formulated.

Teachers need to be prepared to connect their classes with the outside tools like mathematics museums and science exhibitions. This kind of activity needs to be well prepared as teachers deal with completely non-conventional kinds of activities that need to be explored outside the curriculum. I heard horror stories about some student visits to museums when teachers would leave students at ease inside the museum and then the students would literally destroy the exhibits breaking all sorts of buttons, knobs or geometric elements. Maybe a course like *Popularising Mathematics* should be introduced in the initial teacher preparation syllabus.

Cédric Villani, the French Fields Medal winner in 2010, and now a member of the French Parliament, in a recent interview to the channel ARTE said, "It is training that is lacking for teachers of mathematics in France. It is necessary to give them much more chance to experiment, [give] lectures in didactics and pedagogy, [give] many more situations in the fields." (28 March 2018). I completely agree and more systematic plans should be prepared to accomplish this.

## An international Day for Mathematics?

Another very successful popularising initiative is the day dedicated every year to the **number PI**. It was initiated in March 14, 1988 in the *Exploratorium* of San Francisco by the physicist Larry Shaw of the museum staff, including very popular activities namely parades with numbers from the PI decimal places. Now it is very popular in a number of countries including Portugal. There are proposals to UNESCO that it be adopted as an international day for mathematics and this would certainly be a good moment to popularise mathematics in a very strong way.

## Acknowledgement

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# First Language Interference: a guide for teachers of mathematics

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**Keywords: EAL; FE; adult learners; language for learning**

## Introduction

Teaching mathematics in English to classes of students whose first language may not be English is present in many countries including the UK, where the language of education is English, but the population may have varying levels of English competency, as it may be a second, or even third, language.

I teach adults in a further education college in the UK, and in the last few years about 25% of my students are English language learners (ELL), half in discrete ESOL (English for Speakers of Other Languages) Functional Skills Mathematics classes, and half in GCSE Mathematics. GCSEs are generally taken by 16-18 year olds, and are used as gatekeepers for university entrance, hence the need for adults returning to education to gain the necessary grade for an onward journey into teaching or nursing.

The impact of the language content and structure on students' ability to engage with the material in mathematics classes and exams is surprisingly high, and can reduce students' mathematics competencies. This is not a comprehensive guide to all of the differences between English and other languages, but aims to raise awareness of the extent and content of those differences. This is just one barrier to learning that adult students may experience.

## Sources of Information

Initially my research was of a purely observational nature. I became aware that the impact of the English language was having a negative effect on pass rates for ELL students. For instance, a student might come out at Entry Level 3 on a mathematics initial assessment containing few word problems, but struggle to complete an Entry 2 diagnostic, with more word problems, and need to go lower to Entry 1 if their language skills were considerably below those contained in the questions. This observation is confirmed by other practitioners in mathematics classrooms (Kersaint et al., 2013).

This led to a growing realisation of the importance of language demands in mathematics questions, a more language focussed teaching style, and that the resources I was producing for the ESOL learners were appreciated by the English speaking learners, especially those with lower competency levels (Adler, 2001).

Research work in this area led me to a publication by a number of writers with another first language, but high English competencies, about the differences between the languages (Swan and Smith, 2001). This book covers over 20 languages, such as Arabic, Chinese, Polish and Spanish, and some language clusters, such as Dravidian (includes Kannada, Malayalam, Tamil and Telugu), Scandinavian (Danish, Norwegian and Swedish), and West African languages (including Hausa, Ibo and Mende).

There are other works written about the English language that have also proved useful in extending this knowledge (Swan, 1995; Crystal, 1995).

The importance of thought and discussion as aids to learning mathematics has been explored by a number of academics and practitioners, which has led to the view that restricted thought or talk can lead to restricted learning (Sfard, 2008; Barwell, 2009; Woolley, 2013). This has implications for my students: do I encourage thought or talk

in their first language, and then translation into English (Adler, 2001; Kersaint, 2013), or do I insist on discussion in English? If the former, how much does this exclude those learners who do not have anyone who speaks their first language (Newmarch, 2005)?

This led me to view mathematics learning as a triad: procedural competence, conceptual understanding and language learning. I have applied this to all my classes, because language can often be a barrier to understanding and progress for adult learners.

## First Language Interference: Impact on Learners

Two case studies illustrate the actual impact of language differences:

The first is of a learner where the mathematical ability was far greater than was shown by his work in a mathematics class, due to his language level (ESOL Entry 1): he had a Masters degree in mathematics from Poland but just passed an Entry 3 qualification.

The second is a Chinese student's experience of moving to a UK university to continue his studies in Mechanical Engineering, who dropped his degree course because he failed the mathematics exam. In China they do not use Greek letters for formulae, and he was unable to change to the European system in time.

This led to a realisation that the English language, the mathematical language and the symbolism may all impact on how successful learners are when studying mathematics in English (Kersaint et al., 2013; Woolley, 2013). I also need to be aware of the cognitive load on students, as ELL students have a higher load that can impair learning (Ashcraft, 2002). These students may need more time, as they need to translate before they can process the mathematical content (Swan and Smith, 2001).

## Findings of Potential First Language Interference

Some of the many differences between English and other languages, in terms of sentence structure, words and sounds are described before progressing to evaluating mathematical differences (Crystal, 1995; Swan, 1995; Swan and Smith, 2001).

### *Sentence structure, words and sounds*

Written language may not be from left to right, as it is in English, but can be from top to bottom, as in Chinese and Japanese, or from right to left, as in Arabic and Urdu. Punctuation is not present in all languages, for instance in Arabic, Korean and Thai a gap is often left instead. Punctuation use can differ between languages, for instance there is no possessive apostrophe in French, and in German and Scandinavian languages a comma is often placed after the verb in a sentence, but in Turkic languages it is placed after the subject. In Hindi a full stop is a vertical line. In English we use upper case letters at the start of sentences, and for proper names, such as Dublin, Monday and September. In German all nouns are written with an uppercase letter, but in French and Spanish days of the week and months start with letters in lower case, so 'lundi', or 'lunes'. Many languages have no upper and lower case format at all, including Arabic, Chinese or the Sanskrit languages, such as Bengali. In English nouns do not have a gender, but nouns in many languages can be masculine, feminine or neutral, which affects both the spelling and pronunciation of words such as 'the' and 'my', and how adjectives attached to the noun will look. In many non-English languages nouns used as objects in a sentence take their gender with them. Hence you may find that learners will translate, "She has lost her bag", as "She has lost his bag", if 'bag' in their first language is a masculine noun. The order of words within sentences is also a challenge, as it is likely to be different e.g. in Arabic the verb comes first, but in other languages,

such as Sanskrit languages, Tamil and Turkic languages, the verb comes last. The verb 'to be' does not exist in Arabic, Thai, or Turkish. In some languages the use of 'to be' and 'to have' are switched, so learners may say they "I have thirty years old", not "I am thirty years old".The verb 'to do', a common verb in English, does not exist in languages as disparate as the East African languages, French, German, Hindi, Italian, or Urdu. Languages differ in the presence of articles, such as 'the' and 'a'; e.g. 'the' does not exist in Turkish or Urdu, whilst 'a' does not exist in Arabic, Hindi or Urdu. There can be less or different use of prepositions in sentences, e.g. in, on, under, at, for, of, in most languages, including Arabic and Turkish, but these are integral to the lowest level of assessment in England, Entry 1.

Countable and uncountable nouns, and their quantifiers, can also be different in many languages. In English, words such as news, trousers, pyjamas and scissors are always plural, but expenditure, time, money and hundred are always singular. Learners may be confused by the lack of a plural if they see 200 written in words. We have a proliferation of words in English where elsewhere perhaps only one word will be used, including which and what, some and any, much and many. This can add to the complexity of translation for learners of English. Comparatives and superlatives such as taller and tallest, are very common in English, and are in Entry 1 mathematics, but do not occur in languages such as Turkish. The use of questions containing negatives should be used with great caution in class, as different languages have the opposite response to what we would expect in English. If I say, "Are you not going out tonight?" to a native English speaker I would expect a "No" response, but in Thailand the answer would be, "Yes, I'm not going out."

Some words occur in more than one language, for instance 'oval' is the same word in Bulgarian and English, but other words are called false friends: they may look and sound like a word in another language, but they mean something different, e.g. in Scandinavian languages 'rent' means rate of interest, and an 'offer' is a sacrifice. English has a Northern European alphabet, which differs from elsewhere in the world. Letters can make different sounds, even when they look the same, e.g. in Polish the 'e' and 'i' sounds are reversed, as are 'v' and 'w'. Some languages, such as Chinese, German, Spanish and Turkish have no 'w' sound at all, so mispronunciations can occur. English has many vowel and consonant clusters that are not present in other languages. For instance the 'th' sound in English does not occur in many African languages, Arabic or Polish, hence learners may struggle to pronounce thirteen or thirty. The endings of words may not be heard in Chinese or French, but in English it can be crucial to hear these, or the meaning is lost, as in 'fourteen' and 'forty'.

*Mathematical Symbols, Procedures and Terminology*

There are a number of ways in which symbols and mathematical procedures differ between languages, including a variation between the use of the comma (,), and the full stop (.),e.g. three euros and fifteen cents is commonly written as E 3,15 but three pounds and fifteen pence is written £3.15. Also 100.006 will look like 100 point zero, zero, six to English speakers, a decimal number; but one hundred thousand and six to many others.

The division symbol differs in the UK and USA compared to the rest of Europe; in the former it is written as ÷, but in the later it is written :, a symbol used for ratios. In some countries, such as Poland, a dot placed between two numbers is the symbol for multiplication, as it is in higher levels of mathematics in the UK. Rounding and estimation seem not to be present in languages that are pictorial, where one character has a meaning, such as Chinese and Japanese. Changing a number before a calculation

increases the work load, so learners are unused to these practices. Many countries, such as Spain, do not use the 12 hour clock, only the 24 hour version, so learners may make errors in time calculations. There are further differences in time as in many languages, including German, 1330 would translate literally as 'half two'; perhaps this was originally 'half to two'. In English we say numbers are odd or even, but it is much more usual in other languages to say even and not even, as in Italian, 'pari' and 'dispari'. As the word order is different, it can lead to learners reversing the meanings of odd and even. Some words have more than one meaning in English, such as the word 'product', which used to mean an article that was produced, now means something we put on our hair, but in mathematics is multiply. Other challenging words include table, change, expand, factor, modal and chord. Some words have specific mathematical meanings that need to be learned, including integer, indices and circumference.

## Conclusion

There are many ways in which a first language can impact on a learner's success in the classroom, and can interfere in the successful acquisition of English for mathematical purposes. An awareness of the issues and challenges learners face has raised an appreciation of the cognitive load many learners face in a mathematics classroom, where the content is delivered in a language that is not their first language.

A focus on language and definitions can benefit all adult learners, including those who speak English as a first language, because of the amount of specialist vocabulary in mathematics. In mathematical problem solving the context can be critical to comprehension, and this is often dependent on language.

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# Archimedes' best ideas

Paul Stephenson

*The Magic Mathworks Travelling Circus*

**Keywords: physical model; mathematical model; argument by exhaustion**

## Introduction

Inspired by Christopher Zeeman's Christmas lectures in 1978, in 1981 the Royal Institution instituted a programme of mathematics workshops called 'masterclasses', now aimed at able children in two age ranges, Years 5/6 and Years 8/9, and run by centres throughout the UK.

The *Magic Mathworks Travelling Circus* has provided masterclasses over many years. This incorporated charity exists to explore the use of physical models and manipulatives in the learning of mathematics. Corresponding to the real activities, there are animations for the children and notes for the teacher on our website. Our masterclasses consist of experiments by the students, working in pairs or small groups, and demonstrations by the presenter.

A recent addition to our repertoire is *Archimedes' best ideas*, aimed at the senior age range. In fact most of our examples derive from just one idea, the 'law of the lever'. Archimedes took the mathematical model that he found describes the action of a pivoted beam and used it as a heuristic device to tackle problems of mensuration far removed physically from the original instance. Our teaching point is this: it is better to think deeply about a few concepts than superficially about many. Although it is not our primary aim to introduce history into the mathematics classroom, this is a happy consequence of our choice of subject. We use modern terms like 'force' and 'work', but try to make clear that we do not know which Archimedes entertained as concepts. Archimedes knew what Galileo later asserted, that the book of nature is written in the language of mathematics. Whatever he designed for King Hiero of Syracuse existed as a concept to be analysed.

For the delegates at BCME9 we set out 10 stations as a carousel. But we began with a half-hour presentation, summarising what we do in our masterclasses. Here we describe the physical models with their particular teaching points.

1. *A mathematical balance*

There are two variables at play. If the students have not met the device before – or used a seesaw as young children – they may need to make several experiments to find out what's going on before they can enunciate 'the law of the lever'.

Aristotle (or the later author of 'Mechanical problems' who discussed the lever) had noted the ratios we now call *velocity ratio* and *mechanical advantage*, but had not combined them.

2. *A mathematical balance with a water cup on one side*

The students use Archimedes' principle to determine the relative densities of 'gold' and the material composing King Hiero's crown, to discover if the royal goldsmith is a criminal.

The science (1, 2) is statics, the science of forces in equilibrium. Archimedes did not talk about 'forces' but his so-called 'magnitudes' behave like vectors rather than scalars. The science we have now rests upon the twin pillars of 'the

law of the lever' – the principle of equal turning moments – and 'Archimedes' principle' – where we see two opposed forces in equilibrium.

3. *A pin, plumbline and shape cut from card*

The students use the lever to make a *null point* reading to find the centroid.

4. *A pin, plumbline and a card shape that is the profile of the body of water displaced by a 'ship' heeled over at 45°, the wooden 'ship' and a flotation tank*

The students find the relative positions of the centroid of the 'ship' and of the body of water it displaces (the *centre of buoyancy*). According as the former sits left or right of the latter, the ship will right itself or capsize. Following their prediction, the students make the experiment.

The remaining idea Archimedes needed to complete his analysis of hydrostatics is *centroid* or *centre of mass* (3, 4). When Archimedes applied 'the law of the lever' to objects extended in space, he is in effect offering a practical definition of it, but no mathematical definition independent of the concept of turning moments has come down to us.

Source: *On the Equilibrium of Floating Bodies*

5. *A string-&-set square apparatus for drawing a parabola, a flexible mirror strip, a projector delivering a narrow, collimated beam*

The students draw the parabola. They then switch on the beam. One fits the mirror as closely as s/he can to the curve. The other moves the projector parallel to the parabola axis. In every position, the reflected beam passes through the focus.

Did Archimedes set fire to Marcellus' fleet with reflectors in a parabolic array? It is now considered unlikely, but his studies of the parabola paved the way for Apollonius' great work on the conic sections. We meet a sophisticated example at station 10.

6. *A wedge and two rollers; a line ruled on acetate; a nut-and-bolt; an Archimedean screw*

As a lever is a machine, so is an inclined plane. The students physically experience the equation load × load distance = effort × effort distance. The screw is an inclined plane rolled up. The students play with the nut-and-bolt. They then use the Archimedean screw to raise water from a lower tank to a higher tank.

Archimedes may have seen the screw in action on his visits to Alexandria. They may indeed have watered the more ancient gardens of Nineveh (modern Mosul). Although he may not have been the inventor, he certainly described them and put them to work back on his home island of Syracuse, emptying water from the holds of ships. The luxury of hindsight allows us to treat the screw as an inclined plane rolled up. It is not certain that Archimedes saw it that way.

We now think of all the machines Archimedes would have seen in use in Alexandria – pulley blocks, for example, which Archimedes developed - in terms of the mechanical advantage they provide (load/effort) at the expense of their velocity ratio (the relative distances moved by the points of application of effort and load). We now have the concept of work as force × distance moved in the direction of the force, energy as the ability to do work, and use the ratio of mechanical advantage to velocity ratio as a measure of a machine's efficiency in terms of the energy usefully expended

in its operation. Archimedes may not have entertained any of these notions apart from the first two.

7. *Two card hemispheres, approximated by stacks of cone frusta and cones, the one fitting inside a perspex hemisphere, the other, outside*

We show the students Archimedes' way of dealing with an infinite process you never need to complete: the method of exahustion or double contradiction, which Archimedes credits to Eudoxus. The conclusion is that quantity X cannot be greater than L and cannot be less than L. Therefore it must be equal to L, (L in this case being the surface area of a sphere).

Source: *On the Sphere and Cylinder*

8. *An expanded polystyrene sphere in a cylindrical acetate sleeve, pins*

The students insert pins through the vertices of a bent rectangle in the cylinder so that they enter the sphere along radii normal to an axis. They remove the sphere and join the points of entry: the area of the curved shape is that of the bent cylinder.

Source: *On the Sphere and Cylinder*

9. *An acetate hemisphere, sitting in a cylindrical sleeve and pierced by an inverted cone, all of the same height and radius;a vertical cross-section on which the students can write; the same solids that can be filled with sand; a triangular prism dissected into three triangular pyramids of equal volume*

By use of the formula for the area of a circle and Pythagoras' theorem, the students show that, at every height, the cross-section of the hemisphere, and of the space between the cone and the cylinder, have equal area. They then argue by Cavalieri's principle that the volumes of the hemisphere and the space between the cone and the cylinder are equal. According to this principle, if two solids have the same height, and the same cross-sectional area at every height, then they have the same volume. Applying Cavalieri's principle again, this time to the dissected prism, they show that the volume of a cone is one third that of the containing cylinder. From this, the ratio of the volume of the three solids, cylinder:hemisphere:cone, comes out as 3:2:1 (figure 1). Archimedes' heuristic 'method' – so called – involved hanging the solids from the arms of an imaginary balance.

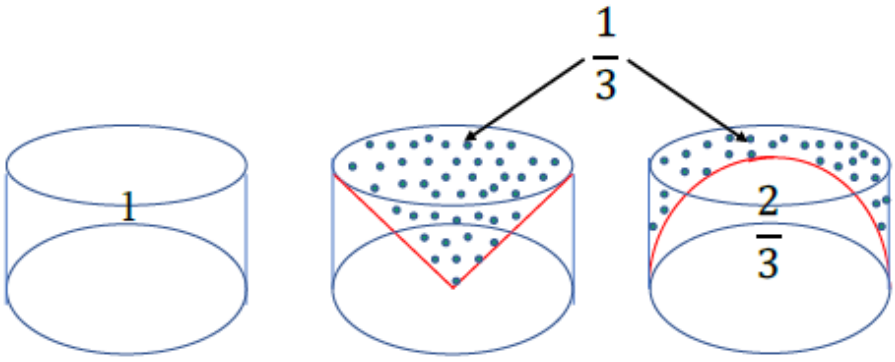



Figure 1. relationship between the volumes of cylinder, cone and hemisphere of equal heights

Source for the cone's volume: Euclid, The Elements, Book XII, Proposition 7. Having conjectured his theorem in this way, Archimedes proved it by appeal to Euclid.

Source: *On the Sphere and Cylinder, The Method of Mechanical Theorems*

*10. A diagram of a parabola cut by a secant, with a tangent drawn parallel, and a triangle drawn with its base on the secant and apex at the point of tangency; successive versions as the space between is filled in the same way*

The students use the triangle area formula 'half base × perpendicular height' to show that each new generation of triangles has $\frac{1}{4}$ the area of the previous set. In terms of the original triangle, they then need to sum the infinite series $1 + \frac{1}{4} + (\frac{1}{4})^2 + (\frac{1}{4})^3 + \cdots$ Two famous proofs-without-words are presented to solve the problem (figure 2). Which do you prefer?

Source: *The Quadrature of the Parabola.*

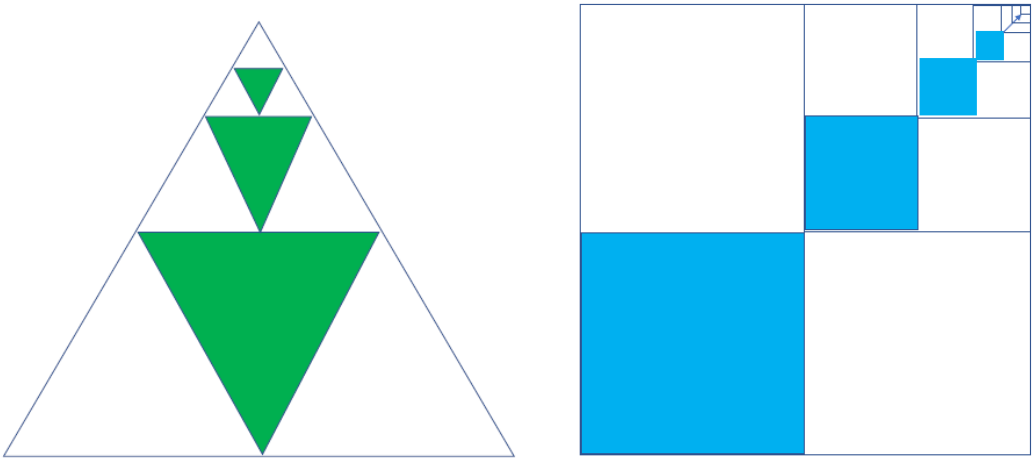


Figure 2. Proof without words for summing the series $1 + \frac{1}{4} + (\frac{1}{4})^2 + (\frac{1}{4})^3 + \cdots$

## Further reading

http://magicmathworks.org/wp-content/uploads/Archimedes-masterclass-session-notes-pdf.pdf

# Lazy Teaching with Active Learners

John Suffolk

*Retired*

**Keywords: practical mathematics; learning by doing; kinaesthetic learning**

Working in countries where there was not much money for equipment, I tried to use learners as teaching aids for mathematics and with help from teachers and colleagues in Brunei and South Africa developed the approaches presented here. I found Lovitt and Clarke (1988) a particularly helpful resource. Learners can be used as teaching aids, giving them a lot of fun in the classroom and teachers much less to prepare before class. Such activities come from many key stages - making areas to find formulas, experiencing being prime numbers, making plane figures to find angle sum formulas, drawing distance-time graphs using learners walking without either rulers or watches, using procedures of bowls for the blind to introduce polar coordinates, quadratic and cubic graphs and loci.

A quarter of a century ago I used to go to classrooms in a comprehensive school in Bristol with piles of textbooks. In Zambia nearly half a century ago I carried equipment a long way to the classroom. Sometimes, conveniently and easily, the learners could be teaching aids, carrying themselves to class.

## Coordinates

Arrange desks neatly in rows and columns. Ask, “How would you tell a friend in another class unambiguously the position of your desk without showing them?” Discussion should lead to agreement that stating the row and column of their desk is sufficient, using the door at the front as the starting point, or 'origin’. The position of desk A in the 3rd column and 4th row, for example, is written as the coordinates (3, 4) with everyone agreeing to write the column number before the row number.

Check that everyone in the class knows the coordinates of their desk. Talk about $x$ coordinates and $y$ coordinates. The axes will be two walls of the classroom.

### *Equations of lines*

Ask all the learners whose $x$ coordinate is 2 to stand up. They make a straight line, whose equation is $x = 2$. Repeat for rows and extend to learners whose coordinates add up to 4, giving a straight line, with equation $x + y = 4$. If the $y$ coordinate is 1 more than their $x$ coordinate they form a straight line $y = x + 1$.

This can be done dramatically with larger groups, perhaps in the school hall. You can use this to show graphical solutions of simultaneous equations, such as $y = 3x - 1$ and $y = x + 3$. Two lines of people stand up and where they meet, $x = 2$ and $y = 5$. If the $y$ axis runs up the middle of the group, you can show other graphs e.g. $y = x^2$.

## Loci

Some examples of the many loci that can be established using learners:

Circles - points equidistant from a point.

Locus of points greater than 3 m but less than 5 m from a fixed point O. A learner is O. Others stand between 3 m and 5 am away. The shape formed is an annulus.

Locus of a points 3 m from a line segment. Some learners form the line. Others stand 3 m from the line and make a ‘running track’.

The perpendicular bisector of line AB. One learner is A, a second is B; others stand uidistant from the two lines.

Points equidistant from two intersecting lines Learners make the two lines. Others stand the same distance from both lines. They form the angle bisector.

These activities are fun as well actively engaging for learners.

**Area of rectangle formula, prime numbers. Is a square a rectangle?**

*Area of a rectangle*

Ask 12 learners to make a rectangular array; they can do this in various ways: 3 by 4 (or 4 by 3 which is the same - commutative law), or 2 by 6 or 1 by 12. Discussion leads to realisation that the number of people is the number in a row multiplied by the number in a column, in other words, area = length × breadth.

Ask one other learner to join the group of 12, now only one array can be made (1×13). Why is this? What other numbers will only make a single array? This helps distinguish *prime numbers* from composite numbers.

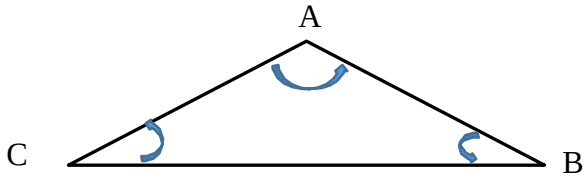
Ask three more learners to join the group of 13, so there are now 16. There may be some debate about whether 4 by 4 is acceptable as a rectangular array. This is to be encouraged as learners need to understand that a square is a rectangle.

Illustrating the distributive law: organise 8 male and 12 female learners thus:

| | | | | |
|---|---|---|---|---|
| M | M | M | M | 4 by 2 males + 4 by 3 females = 4 by 5 learners |
| M | M | M | M | |
| F | F | F | F | $4 \times 2 + 4 \times 3 = 4 \times (2 + 3)$ |
| F | F | F | F | |
| F | F | F | F | |

**Angle sum of triangle, quadrilateral...**

It is quite easy to show the sum of the exterior angles of a polygon is 360°. If you walk around any polygon you will go through all the exterior angles and you will have made a full turn. It is harder to do this with the interior angles. Begin with a triangle made by three learners, A, B and C standing a few metres apart.




Another learner walks around the triangle through each angle in the triangle. Starting on AC, facing A, she walks to A where she turns through the interior angle and maintaining this new heading she walks to B. At B she turns through the interior angle and maintaining this new heading walks to C. At C she turns through the interior angle at C and maintaining this new heading walks back to where she started. She has turned around (i.e she is now facing C), showing the angle sum of a triangle is a half-turn or 180°.

A fourth learner D makes the triangle into quadrilateral ABCD. This time the learner who walks around turning the interior angles ends up facing the same way as as at the start, a complete turn so the angle sum of a quadrilateral is 360°. Repeat for pentagon, hexagon…

## Distance/time graphs without using a ruler or a clock

Invite 10 learners to stand in a line about 1 metre apart. Another learner is the time teller who facing away from the line (so the count is evenly paced) will call out 0, 1, 2, 3, ... at such a pace that as a walker walks along the line she passes one learner on each number. The information can be recorded in a table (spreadsheet) and a graph drawn:

| Time | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| Distance | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

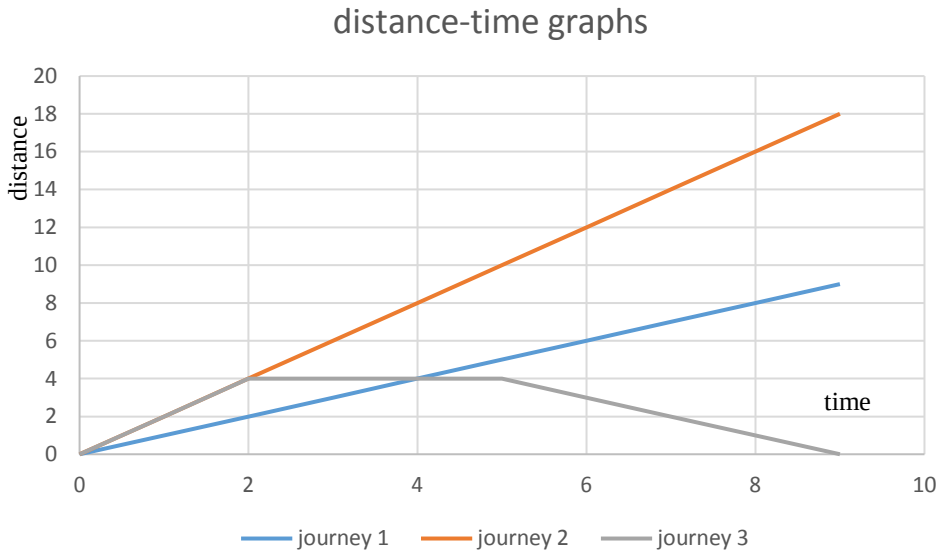
The walker walks again, more quickly this time (so they get to every second person on each number called.

| Time | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| Distance | 0 | 2 | 4 | 6 | 8 | *10* | *12* | *14* | *16* | *18* |

A third journey illustrates the situation when someone goes somewhere, waits and then returns more slowly. The walker walks to the fifth learner in the line. She waits there for a count of three and then returns home. She returns to the first marker more slowly. This is a table for a journey of this kind:

| Time | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| Distance | 0 | 2 | 4 | 4 | 4 | 4 | 3 | 2 | 1 | 0 |

Here are the three graphs on one diagram.

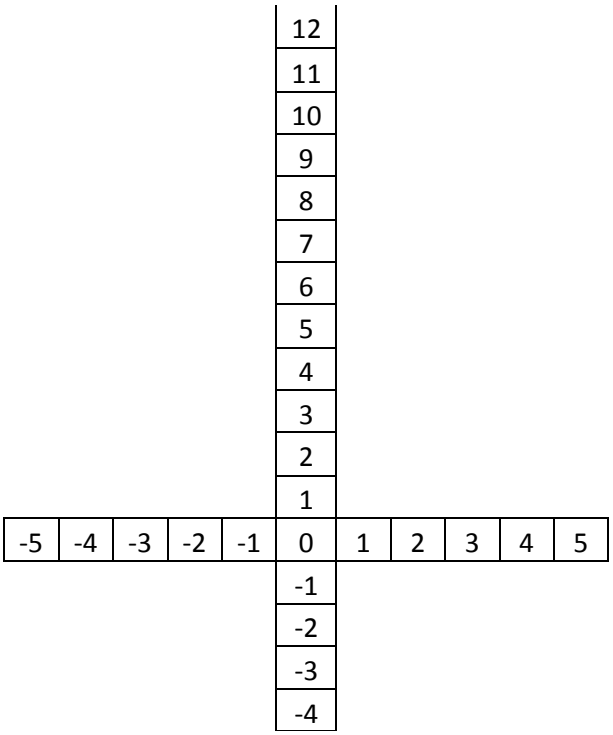



## Graphs

Write whole numbers from - 5 to + 5 and from - 4 to + 15 on large squares of paper and arrange them on the ground as the *x* and *y* axes respectively. Get learners to stand on the *x* axis numbers and then tell them to move parallel to the *y* axis according to instructions.

| 15 |
|---|
| 14 |
| 13 |

(a) 3 paces up (straight $y = 3$)




(b) So $y$ coordinate is 2 more than the $x$ coordinate (straight line $y = x + 2$)
(c) So $y$ coordinate is 3 times the $x$ coordinate (straight line $y = 3x$)
(d) So $y$ coordinate is 2 less than 3 times their $x$ coordinate ($y = 3x - 2$)
(e) So their $y$ coordinate is the square of their $x$ coordinate ($y = x^2$)
(f) So their $y$ coordinate is $(x - 2)(x + 1)$, making a quadratic.
(g) So their $y$ coordinate is $(x - 2)(x - 1)(x + 1)$, making a cubic equation.
(h) And others to show how changing numbers changes shape, position

## Polar co-ordinates and bowls for people who are visually impaired

The aim of bowls is to get as close to the jack as possible. The closest bowl wins. People who are visually impaired have played bowls for many years (see the Visually Impaired Bowls England website). They are helped by a 'marker' who stands beyond the jack and uses a 'clock method' to tell the bowler the position of each bowl relative to the jack. The 'jack' is assumed to be the centre of a clock face. Six o'clock is directly in front and twelve o'clock directly behind the jack, with all other directions given by different times. The player uses the information given by the marker to build a mental picture of the bowls' position.

This approach to defining location can be used to give learners a sense of what it might be like to be visually impaired and to develop an understanding of polar coordinates. The students could role play bowler and marker: the bowler wearing a blindfold whilst bowling, and then sketching the position of their balls (bowls) relative to the jack based on their understanding of the marker's information. They could then compare their sketch with the real situation. The students can make links between the angles, bearings and the times.

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# Fluency through reasoning

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**Keywords: fluency; reasoning; thinking; decision making; noticing**

## Introduction

The aims of the National Curriculum for mathematics identify three key elements: fluency, reasoning and problem-solving. Whilst these are listed individually, it is impossible to consider either fluency or problem-solving without including reasoning. Although some materials suggest teaching the three aims separately this is a misunderstanding of the intention behind the aims and would be counter-productive. This article focuses on the role of reasoning in developing fluency.

## Fluency

Fluency requires decision making and this can be exemplified by considering some calculations. *Let's start with 73 – 69; how would you solve this? Can you draw something to show how you chose to solve it and use this to help you explain why? Now consider 73 – 5; did you choose to solve it the same way? What was behind your decision?*

The majority of people will have chosen to solve these calculations in two different ways; the numbers have been chosen to make this likely. They have been chosen with the expectation that you will notice something which will lead you to make a decision about how to complete each calculation; for example, for 73 – 69 noticing that 73 and 69 are close together or that 69 is nearly 70 or that 74 – 70 is an easier equivalent calculation.

With 73 – 5 you may have noticed that partitioning 5 into 3 and 2 makes this easy to calculate. This thinking could then be extended by asking you to suggest other numbers where subtracting 5 is made easy by partitioning it into 3 and 2, and what you notice about these numbers, then numbers where this is useful for adding 5, adding or subtracting 50 etc. and then numbers where subtracting 5 is NOT made easy by partitioning into 3 and 2 (non-examples). Decision making is at the heart of fluency and includes using what is known to derive, in this case making links between known facts and adding/subtracting in different parts of the number system.

> Fluency is far more than memorisation of facts; research indicates that focusing mainly on memorisation actually works *against* fluency. Countries where high proportions of students rely heavily on memorising, without understanding, perform poorly in international tests (see Boaler and Zoido, 2016).
>
> Understanding is, therefore, an essential part of fluency – in particular, understanding of relationships between numbers and operations, and understanding of the number system (see Russell, 2000).
>
> Babcock LDP Primary Maths Team (2017)

Fluency therefore requires applying understanding as described above, using it to:

- Notice things
- Make connections between what is known and what is unknown
- Make decisions.

There are a number of strategies for supporting fluency through working on noticing things, making connections and making decisions and three of these will be explored here: number talks, target boards and contexts.

## Number talks

Number talks involve presenting students with a problem that they solve mentally. When they have solved it students put up a thumb to indicate they have a solution and then they think of a different way to reach a solution and put out a finger to indicate when they have a second way etc. When everyone has at least one thumb showing the teacher asks everyone to say their solution and all solutions are written on the board. Then different methods and strategies are explored and links are made between them in order to draw attention to structures and relationships. This might include representing the mathematics in different ways such as with symbols and with drawings.

For example, consider how you would solve the calculation 17 × 9. Then, think of another way it could be solved. In a classroom there might be a number of solutions offered and these would all be written on the board and then people invited to explain, or defend, their solution.

One student, Sam, says he has the solution 153 and describes partitioning 17 into 10 and 7 and multiplying each by 9. The teacher has a number of decisions to make at this point starting with how they will record the calculations described by the student. In this case it could be:

Sam $10 \times 9 = 90$
$7 \times 9 = 63$
$90 + 63 = 153$

Or

Sam $10 \times 9 + 7 \times 9 = 153$

The teacher then invites everyone to draw a rectangle representing 17 × 9 and use it to explain what Sam did and why this method works (figure 1).

Explicit links can be made between the verbal representation, the symbolic representation and the pictorial representation of the mathematics; for example asking *Where is 7 × 9 in your drawing?* and *Which part of the calculation does this part of your drawing show?* Sam's method provides an opportunity to explore the distributive law with an image that shows why it works.

A second student, Tegan, explains how she multiplied 17 by 10 and then subtracted 17. Again the teacher has a choice about how they record this:

Tegan $17 \times 10 = 170$
$170 – 17 = 153$

Or

Tegan $17 \times 10 – 17 = 153$

Everyone is again invited to draw a rectangle to show why this works (figure 2).

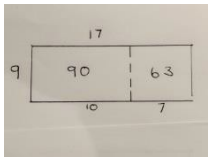



Figure 1. Sam's method

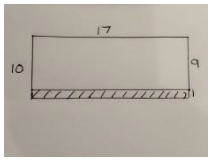



Figure 2. Tegan's method

The teacher then asks everyone to compare Sam's and Tegan's methods identifying what is the same and what is different. This is why the methods are named,

as well as valuing the individual's contribution, it is so that they can be directly referred to in discussion.

Thinking could now be further extended by asking students to suggest a context for the multiplication or to suggest another pair of numbers that it would be easy to multiply using Sam's method or using Tegan's method.

There are a number of key aspects to number talks including:

Numbers need to be chosen for particular reasons, e.g.

- provoke awareness of structure
- expose mathematical ideas and connections
- encourage discussion of efficiency

The focus is on thinking, students are expected to:

- notice things and use what they know
- work mentally to find a solution
- think in more than one way to find a solution

Number talks provide an opportunity to:

- make sense of different ways of solving the problem
- compare solutions
- represent methods in different ways

## Target boards

Target boards can be used to support students to make decisions based on what they notice, making connections between what they know and what they need to find out. They can be used with different ages. The main decision for the teacher is the numbers that will be included to support fluency.

An example of this might involve supporting students to use what they know about the additive composition of seven to add and subtract with three-digit numbers. To start with students might be asked to write down all the pairs of whole numbers that make seven. They are then shown the target board (figure 3).

| | | | |
|---|---|---|---|
| 546 | 325 | 562 | 273 |
| 834 | 903 | 471 | 135 |
| 562 | 731 | 699 | 872 |
| 181 | 254 | 436 | 953 |

Figure 3. Target board for adding and subtracting with three-digit numbers

Students could be asked to choose a number from the board, write it down and then subtract seven, thinking of a realistic context for this calculation. Discussion would follow linking this to the additive composition of seven for example asking: *Did anyone use the fact 7 = 5 + 2? Which number do you think they chose? Why? Could it be another number? Is the context realistic for these numbers? Did anyone not use any of the facts for seven we wrote down? Which number do you think they chose? Why? What if you add seven to your number, will you use the same fact? Can you write a three digit number where splitting seven into three and four is useful when adding seven? Can you write a three-digit number where adding seven does* ***not*** *involve splitting seven?*

Alternatively, the students could be told that splitting seven in different ways is useful for subtracting seven from all of the numbers on the board except one and

then ask them to find that one and to explain why splitting seven is not needed when subtracting seven. They could then create their own board for adding seven where splitting seven is not useful for only one of the numbers.

## Contexts

One aspect of fluency is recognising familiar mathematics in unfamiliar situations. One way to approach this is to start from contexts, presenting students with a calculation and a number of scenarios and asking them to decide which fit the calculation, which don't and why. For example, for the calculation 80 ÷ 10 = 8

> I bought ten packets of sweets with eighty sweets in each packet.
>
> Chairs in the school hall are set out in rows of ten. There are eighty people to seat.
>
> The cook made eighty biscuits and had ten bags to put them in for selling.
>
> I saved up eighty pounds and then spent ten pounds on just one new dress.
>
> Babcock LDP Primary Maths Team (2017)

Discussion could then focus on the meaning of the numbers in the different scenarios that match, for example: *What does the eight represent in the second scenario?* Students could write calculations to match any scenarios that do not fit the given calculation and then create their own set of scenarios for a calculation, where some fit and some don't but the idea is to make them all seem like they could fit.

## Conclusion

> *The* true test of intelligence is *not* how much *we know* how *to do*, but how *we* behave when *we* don't *know* what *to do*. (Holt and Fromme, 1964: 163)

Mathematicians work on problems to which they do not have the answers. In order to do this they notice things and make connections between what they know and what they are trying to find out. We need our students to do the same and to be aware of the need to make decisions in mathematics. Fluency is dependent on decision making. Drawing attention to decisions, discussing decisions, demanding justification of decisions and making connections with other situations where the same decision would be a good one or would be a bad one are all ways to support an understanding of the importance of making decisions in mathematics. Number talks, target boards and contexts all provide opportunities for focussing on decisions.

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# Concept-Based Mathematics: Teaching for Deep Understanding

Jennifer Chang Wathall

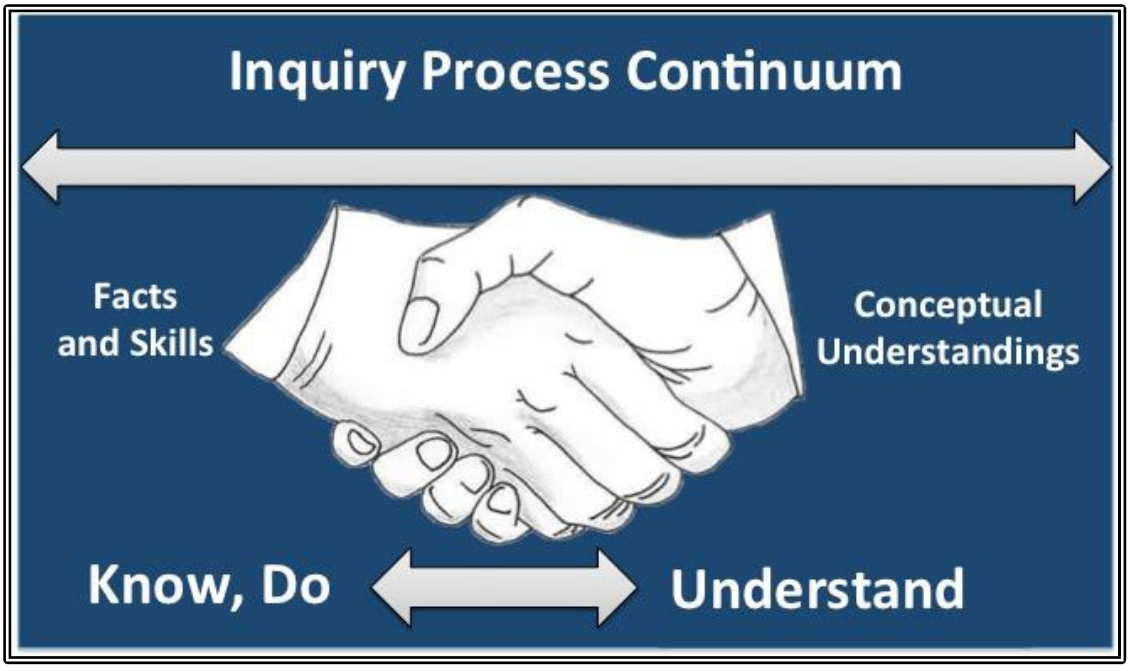
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**Keywords: concept-based; inquiry-based learning; understanding**

Traditional curriculum focuses on rules and procedures with little understanding of the conceptual relationships of mathematics - and mathematics is a language of conceptual relationships. Traditional curriculum assumes the deep understanding of concepts, and fails to teach for transferability, or to consider context. Concept-based mathematics expands on the work of Erickson (2007) and Lanning (2014) into the realm of mathematics. Specifically, concept-based mathematics extends their work on the Structure of Knowledge and the Structure of Process for the mathematics realm to help educators understand how to convey mathematical concepts and ideas by utilising the vehicle of inquiry.

When learning mathematics, students need to be given time and space to explore and discover the beauty and creativity in mathematics without being fearful of mistakes. Maths Anxiety exists because of an over emphasis on the processes and skills of this discipline. The concept-based model addresses how to create a concept-based and inquiry-led curriculum and instruction with a goal to make mathematics enjoyable and accessible to all of our students.

A concept-based curriculum is a three-dimensional design model of curriculum and instruction that frames factual content and skills and processes with disciplinary concepts, generalisations and principles. In a concept-based curriculum and instruction the development of intellect is achieved through higher order, synergistic thinking where teachers use the facts, processes and skills in concert with the concepts, generalisations and principles. Figure 1 illustrates how to develop intellect in students.




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Figure 1 Developing Intellect utilising the Inquiry Process Continuum Model

A traditional two-dimensional design model for curriculum and instruction focuses on factual content, processes and skills and assumes conceptual understanding.

Gao and Bao (2012) reported a study with 256 college level calculus students exploring concept-based instruction. The findings show that students who were enrolled in the concept-based learning environments scored higher than the students enrolled in the traditional-learning environment. Students in the concept-based learning courses also liked the approaches more. A better grasp of the concepts involved results in increased understanding and transferability. The research and consensus on the benefits of developing conceptual understanding is undeniable. A concept-based curriculum produces deeper emotional and intellectual engagement in learning and therefore develops attributes such as critical thinking, reasoning and creativity.

In this ever changing, dynamic and complex world, mathematics education must engage students intellectually and emotionally. The ability to think conceptually, transfer understandings across contexts and situations, and to enjoy learning and problem solving are major goals for mathematics education today so we can prepare our students for future success. Technological advancements of even the last decade have influenced instruction and key to utilising technology effectively is not "what" tool is being used but "how" the technology is used to enhance learning.

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# Challenging the fear: a framework for addressing anxiety in adults learning mathematics

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**Keywords: mathematics; attitudes; perceptions; confidence; understanding; teaching**

## Introduction

Having identified a trend in student anxiety related to learning mathematics when embarking on undergraduate education studies, the early stages of my doctoral research focussed on exploring the views of a cohort of 75 first year undergraduate students who were attending a part-time BA (Hons) Applied Education Studies degree (Wicks, 2014). Students completed questionnaires before and after their first undergraduate mathematics unit to identify their perceptions of the subject. Analysis of the questionnaires indicated that on completion of the units, students identified increased levels of confidence and understanding in learning mathematics, and a more positive disposition towards the subject. Also identified were a range of factors affecting the students' learning, and these included the effect of the teacher and teaching strategies, personal perceptions, setting arrangements (pre-teaching questionnaire only) and the role of discussion.

## Perceptions related to learning and teaching mathematics

Concerns regarding adults' perceptions related to learning mathematics have been previously established, with Richardson and Suinn (1972) suggesting that mathematics anxiety can interfere with the process of manipulating numbers and solving problems. Others have expressed similar concerns, identifying that anxiety can limit the ability of those affected in completing mathematical activities (Ashcraft and Krause, 2007; Boaler, 2009; Evans, 2002; Tobias, 1993). Further exploration of such anxieties provides a concern that where teachers are anxious about learning mathematics themselves, this may have an effect on their teaching of the subject, particularly at primary level (Ball, 1990; Relich, 1996). In particular, there is a suggestion that those who are anxious about mathematics themselves may pass this on to the pupils they teach (Bekdemir, 2010; Brady and Bowd, 2005; Haylock, 2010).

With the potential that anxiety in learning mathematics may be passed on to students, and pupils in the UK performing less favourably in comparison to international counterparts (OECD, 2016), further consideration needs to be given to how to support adults who are anxious about learning mathematics – particularly those working with children in schools. Coben (2006) suggests that there is limited consideration of how to support adults in learning mathematics, but there are those who have considered how adults might learn and could essentially provide a structure for this. In particular, Knowles et al. (2005) built on the work of Lindeman (1926), offer an andragogical model for adult learning. Their six assumptions are based on the idea that adults need to be prepared to learn, rather than just focussed on content, and that they need to be supported in making connections between one aspect of learning and another.

In order to establish potential strategies to support adults learning mathematics, Knowles et al.'s six assumptions for adult learning will form part of the theoretical base for analysis of data discussed within this paper.

## Methodology

The aim of the research was to identify strategies to support adults learning mathematics on an undergraduate education degree and a mixed methods approach was taken that involved collection of both quantitative and qualitative data. Survey research was used to explore the students' perceptions before and after their first mathematics education unit, the results of which have already been previously been shared (Wicks, 2014). In order to probe more deeply into the students' perceptions, focus group discussions were held with ten students. Four activities were designed to engage discussion about factors that may have affected their learning of mathematics and identification of strategies to support them in this process. These included an exploration of language associated with mathematics to identify feelings towards the subject; a ranking activity designed to explore the range of influences on the students' learning; and two discussion activities to probe more deeply into the role of the teacher, teaching, discussion and personal perceptions identified within the student questionnaires. Newby (2010) suggests that themes (or templates) can be used to examine views in depth and this is the approach used within the analysis of the focus group discussions and triangulation of the data.

The sample of students chosen for the full research study was 75 first year undergraduate students enrolled on a part-time BA Applied Education Studies degree in 2011/12. The final phase of the research was to identify students for whom there had been a potential change in perceptions and students who had made the greatest rates of progress, from their initial mathematics test audit (June 2011) and the mathematics tests associated with the first unit (March 2012), were invited to participate. The rationale behind this was rooted within the findings of those identifying a positive correlation between confidence and performance (Ashcraft and Krause, 2007). Of the 36 students who were invited, ten chose to take part in the focus group discussions.

## Results

The results of Activity 1 demonstrated that the students were more positive towards mathematics than they had been previously. The ratio of positive:negative vocabulary identified was 2:1 and alongside this they identified that in the main they felt more confident than they had before.

Activity 2 involved students ranking a range of potential influences and Table 1 identifies the top three influences for each of the three focus groups.

| | **Focus Group 1** | **Focus Group 2** | **Focus Group 3** |
|---|---|---|---|
| 1 | Teacher<br>Teaching | Attendance at sessions | Teacher<br>Teaching |
| 2 | In-class discussion<br>Other students<br>Attendance at sessions | Teacher<br>Teaching | Online materials<br>Websites |
| 3 | Online materials<br>Outside influences | Online materials | In-class discussion<br>Other students<br>Attendance at sessions |

Table 1: Summary of the top influences in learning mathematics as adults

Activity 3 focussed on examining factors related to teaching, the only element identified by all students within the post-teaching questionnaire. Characteristics of the teacher which supported them in feeling comfortable included patience, understanding and approachability. Students also identified a range of teaching strategies that had aided their confidence and understanding including: clarity of demonstrations and explanations, breaking down mathematics into understandable parts, having a suitable pace, allowing for opportunities to discuss and ask questions, and having time to practise.

Activity 4 allowed the students to discuss additional influencing factors related to discussion with others and personal perceptions. Students acknowledged that working with others could be supportive in developing understanding, but only if peers were like-minded. Discussions related to personal perceptions acknowledged that there were a range of positive and negative perceptions related to learning mathematics, and some surprise that students knew more than they originally thought.

## Discussion and implications

The aim of the research was to identify strategies that may support adults learning mathematics on an undergraduate education degree. The results of the post-teaching questionnaires and focus group discussions demonstrated three key influences related to the teaching of mathematics: the teacher and teaching, personal perceptions, and the role of discussion; with factors related to the teacher and teaching identified as the highest positive influence. Closer analysis of the data has supported the identification of seven teaching strategies that might be considered when supporting adults learning mathematics. These have been constructed alongside Knowles et al.'s (2005) six assumptions for adult learning. Table 2 identifies the seven proposed teaching strategies alongside these assumptions.

| Teaching Strategy (TS) | Assumption |
|---|---|
| TS1: Clear modelling and explanation of strategies<br>TS2: Break down mathematics to show how each area is developed | The need to know<br>Motivation |
| TS3: Make connections between different aspects of mathematics | The role of experience |
| TS4: Have a pace appropriate to the level of students | Readiness to learn |
| TS5: Allow time for questioning and discussion<br>TS6: Provide time for practise within sessions | Readiness to learn<br>Orientation to learning |
| TS7: Provide (online) practise materials outside of teaching sessions | The learner's self-concept. |

Table 2: Identification of seven proposed teaching strategies to support adults learning mathematics

### *Implications*

The findings from the study have resulted in seven proposed teaching strategies that might be considered in supporting adults learning mathematics. However, it is also acknowledged that other factors may also play a part, including the personal characteristics of the teacher, personal perceptions and the role of others. The results of the study and the application of these proposed teaching strategies now need to be explored through further research and across a wider range of experiences involving adults learning mathematics.

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# Topics taught by trainee secondary mathematics teachers

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**Keywords: teacher education; lesson aims and objectives**

At the end of the 2016/17 academic year, 26 (out of a cohort of 32) trainee secondary mathematics teachers handed in the lesson plans they had produced and stored in electronic form during their training year. The intention was to log this data set by lesson aim/objectives, and obtain an overview of the areas of the mathematics curriculum that were being taught during the school placements. Consideration could then be given to what additional training experiences, if any, might be introduced in relation to any areas of apparent neglect. My original concern was that the trainees were teaching very few statistics lessons and I intended (a process still underway) to quantify the extent to which this impression was indeed the case.

## *Data from analysis of the lesson plans*

The data set totalled 1530 lesson plans from the 26 trainees, of which 278 have been recorded against statements taken from the Mathematics Programmes of Study (PoS) (DfE, 2013 and 2014), five 'revision' lessons were removed from the data as they could not be mapped to any particular PoS statements. Of these, 15 lessons have been recorded against the 26 statements within the Working Mathematically sections of the PoS: 10 in KS3 (five of the 18 statements have lessons recorded against them), five in KS4 (two of the 18 statements having lessons recorded against them). The intention is to ascertain the extent of trainees' experience at teaching Working Mathematically related lessons before reallocating these to the subject content statements. Of the remaining 263 lessons, table 1 shows the distribution by Subject Content area and key stage:

Table1

| | Number | Algebra | Ratio, Proportion & Rates of Change | Geometry & Measures | Probability | Statistics | % |
|---|---|---|---|---|---|---|---|
| KS3 | 46 | 70 | 29 | 35 | 8 | 11 | 76 |
| KS4 | 2 | 24 | 10 | 21 | 6 | 1 | 24 |
| % | 18 | 36 | 15 | 21 | 5 | 5 | 100 |

One notable interim finding relates to curriculum coverage. In KS3, lessons have been recorded against 10 of the 16 Number PoS statements, one of these ("use the four operations, including formal written methods, applied to integers, decimals, proper and improper fractions, and mixed numbers, all both positive and negative" (DfE 2013: 6)) having 19 of the 46 examples so far recorded. There is also one KS3 Algebra PoS statement, "simplify and manipulate algebraic expressions to maintain equivalence by… multiplying a single term over a bracket" (DfE 2013: 8), which has 31 examples from the lessons recorded. These two statements (from around 150 statements in the combined KS3/4 PoS) account for almost a fifth of the lessons recorded. Additionally, of the 10 KS4 Ratio, Proportion and Rates of Change lessons, one (of six) statements ("compare lengths, areas and volumes using ratio notation and/or scale factors; make

links to similarity (including trigonometric ratios)" (DfE 2014: 9)) was recorded nine times.

With fewer statistics lessons recorded at this stage than any of the other mathematics curriculum areas, and only one of the 64 KS4 lessons teaching statistics, concerns raised prior to the study are being realised. A further concern is developing alongside this, there being a distinct possibility that a small number of PoS statements are being prioritised, resulting in a far narrower experience for our trainees of teaching a range of mathematical topics on placement. With data recording underway but far from complete, definitive conclusions will have to wait.

What follows here is a summary of discussions undertaken during the BCME session that touch on this, and follow-up on two other emerging issues. I am grateful to BCME and the participants at my session to have had the opportunity to raise and discuss this at the conference.

### *"Get rid of the brackets"*

As I was recording lesson aims and objectives the range of approaches trainees were using intrigued me. I am sure we have all heard the phrase "get rid of the brackets" at some point; many of us will have used it. I doubt that a bracket-free world, certainly not a mathematical one, counts as an 'aim' or 'objective' of teaching mathematics. In a lesson plan from November 2016 one of the trainees had nonetheless written this phrase as a lesson objective. A few months later (March 2017) the same trainee used "rewrite expressions with brackets removed" for their lesson objective. I recorded both of these as examples of the *Mathematics programmes of study: key stage 3 (KS3)* statement "simplify and manipulate algebraic expressions to maintain equivalence by… multiplying a single term over a bracket" (DfE 2013: 8). As mentioned earlier, this is the statement I find I am recording most frequently, so perhaps not surprisingly that there would at some point be more unorthodox examples coming to light. The reasons for this variety can at this separation in time be mere speculation, but there are many other examples that contrast a less formal style with something more closely related to the formal statements of the PoS. The trainee's worthy intention might have been to express the mathematics to be learnt in a way that these students might find familiar and/or accessible. A few months later, with a different class in a different school, the trainee's lesson objective – "rewrite expressions with brackets removed" – now bears comparison with the corresponding statement from the NC, with the trainee using "rewrite" in place of "simplify and manipulate . . . to maintain equivalence", "expressions" for "algebraic expressions" and "brackets removed" as an alternative for "multiplying a single term over a bracket". During the BCME session we discussed possible benefits of the trainee's wording compared to that in the PoS statement, saw little improvement and plenty to regret in the absence of "maintain equivalence". Absent from our deliberations was any reference to the schemes of work from the two schools where these lessons took place, or information on any published scheme that might have been consulted (I do not have access to them). However, whatever possible source of alternative wording might have guided this choice of wording for the lesson objective, there are requirements for Qualified Teacher Status (QTS) and the *Teachers' Standards* (DfE, 2011) to consider. In particular, Teachers' Standard (TS) 3 requires trainees to "demonstrate good subject and curriculum knowledge" (DfE, 2011: 11). This prioritises familiarity with the National Curriculum programmes of study over any particular published resource or school's scheme of work. There was some agreement that in future the use of National Curriculum wording when planning lessons might

prove a beneficial requirement for secondary mathematics trainee teachersThe previous example was taken from the Subject Content section of the PoS, after which discussion focused on the Working Mathematically section. This is subdivided in both KS3 and KS4 PoS into *Reason Mathematically*, *Develop Fluency* and *Solve Problems*. Having discussed participants' personal teaching experience of these areas, mention was made of several Subject Content statements that included problem solving, there being little apparent reason why these could not be located within the Solve Problems section of Working Mathematically.

Solving problems is the most frequent Working Mathematically section referenced by trainees within their lesson plans. One year 7 lesson aim from March 2017 was presented during the session as typical of the trainees' aims/objectives that relate to problem solving: "To be able to solve word problems involving algebra". The same trainee planning a year 9 lesson (May 2017) specified, "To derive equations from worded problems and solve them" as the lesson objective. These examples expose the influence of examinations on the mathematics curriculum taught in both KS3 and KS4. There appears to be a widespread link being made to treat problem-solving as the 'interpretation' of worded problems symbolically, resulting in the formation of an equation to be solved. Whilst consistent with some aspects of the PoS's 'Solve Problems' requirements, the possibility that examinable types of problems are emerging was raised. There is a risk that whatt 'problem-solving' can be assessed becomes what is taught, thus narrowing both the learner experience, as well as the range of mathematics teaching trainees experience.

Consequent to this discussion, the use of 'model' within the PoS was briefly considered, and interpretations of 'interpret' within both the PoS and trainees' lesson plans were raised. One example seen is a trainee's lesson objective stating, "recognise m & c as representing gradient and intercept". Is recognition of m as the gradient distinct from or similar to "interpret the gradient of a straight line graph as a rate of change" (DfE, 2014: 9)? If distinct, then the possibility of a more specific reference to the programmes of study might have prompted a rethink on that lesson's possibilities. Two other examples of 'interpret' were looked at: from KS4 "set up, solve and interpret the answers in growth and decay problems . . ." (DfE, 2014: 9), and from KS3 "interpret when the structure of a numerical problem requires additive, multiplicative or proportional reasoning" (DfE, 2013: 4). For each of these a distinct understanding of 'interpret' is required, perhaps recontextualise, deduce, identify. Having indicated the possibility of undertaking further study into how mathematical activity is described, the potential benefits of asking trainees to expand on the widespread current practice of including mathematical nouns amongst a lesson's key vocabulary and to also include terminology describing mathematical activity was considered. This would be a possibility from year 7, where teaching percentages includes "interpret percentages and percentage changes as a fraction or a decimal, interpret these multiplicatively" and "interpret fractions and percentages as operators" (DfE 2013: 5) are topics commonly taught by trainees.

### *Future training*

The session closed by revisiting lesson aims and objectives. One possibility I am considering is to utilise the standardised lesson plan format of an aim section followed by an objectives one, required of all Secondary PGCE subjects where I work, to provide a 'Working Mathematically' aim followed by 'Subject Content' objectives. A suitable example of this, one discussed during the session, from the KS3 Develop Fluency

statements might be to "develop algebraic… fluency" (DfE, 2013: 4), with a subject content objective "multiplying a single term over a bracket".

Finally, from the trainees' perspective, of paramount importance is the benefit these embryonic developments offer to them in relation to their pedagogical subject knowledge development, and the evidence that this provides for demonstrating TS3. Without placing emphasis on the Working Mathematically sections of the mathematics programmes of study, it is not possible for trainees to "have a secure knowledge of the relevant subject" (DfE, 2011: 3). In relating school and published schemes of work to the mathematics programmes of study, adapting the latter in light of familiarity with the former, trainees can "demonstrate a critical understanding of developments in the subject and curriculum areas, and promote the value of scholarship" (DfE, 2011: 3).

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# Whispering Away Maths Anxiety

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**Keywords: maths anxiety; confidence; KS3; self-reflection**

## Introduction

On hearing the term horse whispering strategies to use in mathematics education, most people recall their most tricky student and wonder how they may 'blow into their noses'. This thankfully is not required here, instead we are going to consider basic strategies of horse whispering and how we can apply them to help students who suffer from maths anxiety.

Conventional equestrian training uses a simple approach of inflicting pain to encourage the horse to do as requested and removing the pain, when the horse obliges. Having worked as a professional equestrian trainer for 30 years, I realised this may not be the most effective method of training horses. Changing careers and now teaching mathematics in a mainstream secondary school, I decided that my experience of interacting and teaching an animal may provide useful in helping my human students.

## Horse whispering in the mathematics classroom



Figure 1. Horses look for a good herd leader

Horse whispering is based on fulfilling the horse's basic need of having a good herd leader that will keep the individual horse safe (figure 1). As teachers we have a similar role. Horse whispering consists of three basic aspects, described below.

### *Setting up a comfort zone*

For the horse this means setting up a comfort zone for them around us. They may not encroach on our personal space and must feel free to leave as they wish. We stage manage this however, so if they try to leave, this becomes an uncomfortable option, resulting in their sudden loss of balance. With careful planning this results in the horse adopting us as its herd leader, choosing to follow our instruction, since it wants to be 'in our club'.

As a fairly 'new' mathematics teacher I quickly realised that students with maths anxiety do not feel that they belong to 'the maths club'. Instead they feel that everyone in this club simply knows things they will never know. I decided that if I could communicate with a horse in such a way as to make it choose to be in my 'club', it should be possible to do the same with my mathematics students.

How could we set up a comfort zone in our mathematics classroom? For me it is about exploring students' view of the world, fulfilling their need to learn, as well as finding relevant applications for the mathematics taught. For this it is essential to have personal conversations with every student and establish a culture of 'we're all learners'.

I myself have poor working and short-term memory, so I decided to use Steve Chinn's tests of both types of memory in my classes, publicly measuring my own memories (Chinn, 2012). I expected to and came out about average for each class,

which caused great excitement amongst the students. It facilitated discussions on what good or poor memories might mean, as well as how these may be affected by noise.

I further set a up a Growth Mindset culture, using Jo Boaler's week of inspirational mathematics at the beginning of the academic year. This was using Dweck's idea that adopting a growth mindset has a positive effect on learning (Dweck, 2012). It was important to me that every student became aware of their own learning needs, to enable them to find strategies which would work for them as individuals.

### *Allowing free choice or at least appearing to do so*

Horses are animals of flight, so it is essential that they do not feel restrained, for them to be able to learn. An example of this is teaching a horse to load. A conventional approach is to attack a horse to force it to go into a waggon. This results in some horses feeling that for an unknown reason, humans become very aggressive whenever the horse comes near a waggon, making lorries best to be avoided at all cost. A more sensitive approach asks the horse to quietly move forward, and if it refuses, re-establishing the comfort zone, making the horse choose between being allowed to 'stay in the club' and following the instruction to load or having to leave the club.

For our learners this free choice is allowing them to choose the degree of difficulty at which to work. This is monitored, but in my experience only rarely needs adjustments. The most likely adjustment needed is students choosing work which is too easy and then commenting that it was too easy. I always comment on this and explain the need for challenge if learning is to take place. This is a culture that can take time to establish but teaches resilience and supports students who are struggling to make sense of new concepts.

Another way of allowing free choice is to use daily reflection by students. My students all complete a 'what went well, even better if' type of self-reflection at the end of every lesson. Again, this can take time to habitualise. We think in detail about what has been learned in the lesson to reinforce connections in the brain and support retention. This reflection is followed by a written response by the teacher after every lesson. This is not as time consuming as many think, often only taking 10 minutes per full class. Research has shown that confidential self-reflection is far more accurate and therefore effective than public shows of self-assessment (Cambridge Mathematics, 2017). My approach goes further, in that I encourage students to write me messages in their books about their work (see figures 2 and 3). Many do this, and it often opens a useful dialogue with individual students. In addition it is highly informative for my short-term planning. If most of the class indicate they need more time on a topic, this is accommodated or if only individual students need more time, support can be given whilst the others are working on a starter task at the beginning of the next lesson.

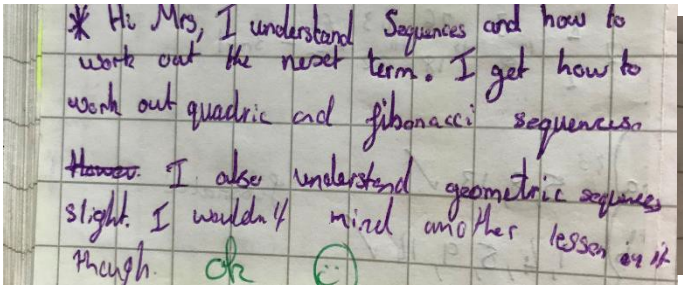



Figure 2. Student self-reflection, initial response

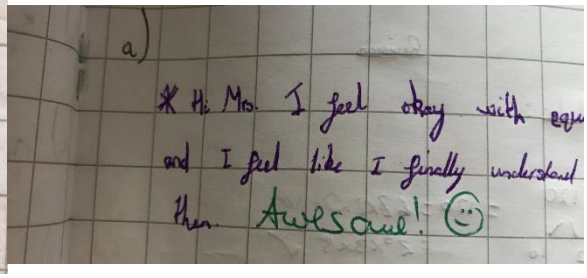



Figure 3. Student self-reflection, final response

Where previously I would have been faced with books full of ticks, I now have very clear indications of how my students feel about their work. My feedback most often is only in the form of 'I agree' or a more individual comment relating to their progress. The response by the teacher after every lesson is very important to students, making them feel appreciated and safe, since they can easily note down any concerns they have. It is every effective in counteracting that feeling of 'not being in the club' of mathematicians.

Returning to the example of horses learning to load. Once a horse is easily following the command to go into the waggon, I would always allow it to rest in its stable for a while, before repeating the same process. A tiny number of horses had fully learned to follow the command to enter the waggon, however most would need renewed practise for 10 minutes or so. A small number would appear to have never entered a waggon before, needing to be taken through the whole process again, albeit for far less time than on the first attempt. Only subsequent attempts made it clear to me what each horse's needs for repetition were.

The same applies to mathematics students. I help my students to explore their own need for repetition, making them aware and therefore enabling them to find strategies for their own learning. This process also involves the categorisation of mistakes, making all students aware of the value of making mistakes. Many students who lack confidence in their mathematics ability have a good grasp of concepts but continually make 'silly number mistakes', which then make them feel that they 'just don't get it'. It is often highly liberating for them to realise the type of mistakes they are making. Calling them 'silly number mistakes' initially identifies them as of less concern. Later on, however, when confidence has improved, the need for accurate number skills is highlighted.

***Rewarding the right choices***



Figure 4. Rewarding a horse making the right choice

With horses this must be done within 0.2 seconds of their right choice, or they do not connect the reward with the choice made (figure 4). A simple example is: horses are often kicked with the rider's lower leg to make them move more quickly. A pressure is applied, and the horse gives a variety of responses. Hopefully one of these is to move faster. If the pressure is removed as soon the horse decides to move faster, it will acknowledge this response as being the correct choice to make. Unfortunately for most horses, many riders do not appreciate this and continually kick the horse on. As far as horses are concerned, continual application of this pressure obviously must mean that moving faster was the wrong answer.

In mathematics education, correctly timed application of reward equally has a significant impact on progress. I point out improvements in progress to my students,

whenever I notice them, particularly after tests, even if the test mark has not improved, any improvement in working I find, is rewarded. Anxious students need help to identify what they are doing well. This further allows me to clearly link their progress to choices made, both good and bad.

Linking lack of progress in tests to short term and working memory equally allows me to facilitate strategies for students, making the results less personally damaging to them as individuals. It creates a distance between their own identity as mathematicians and their ability to store and recall facts. Students with severe anxiety benefit greatly from teachers noticing and pointing out all tiny aspects of good practice, allowing them to build up a picture of 'what they can do'.

## Conclusion

In conclusion, horse whispering techniques allow us to help anxious mathematics students find their place in the 'maths club', reflecting on what works for them and building strategies to enable them to progress more easily.

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## Images

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Figure 4: http://gallery.horsedeals.com.au/index.php?p=library-item&id=49 (accessed May 2018)