Mathematics Assessment Project

The University of Nottingham

UNITED KINGDOM · CHINA · MALAYSIA

Classroom Challenges for Formative Assessment

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Centre for Research in Mathematics Education

University of Nottingham

NANAMIC 2013

What is a formative assessment lesson?

I'm calling on our nation's governors and state education chiefs to develop standards and assessments that don't simply measure whether students can fill in a bubble on a test, but whether they possess 21st Century skills like **problem solving** and **critical thinking** and **entrepreneurship** and **creativity**.

Remarks to the Hispanic Chamber of Commerce on a complete and competitive American Education March 10 2009.

COMMON CORE STATE STANDARDS FOR

Mathematics

Goals of Mathematics Teaching

- **Procedural Fluency**
  - *carrying out procedures accurately and fluently*
- **Mathematical Understanding**
  - *the ability to justify why a particular mathematical statement is true or where a mathematical rule comes from.*
- **Mathematical Processes / Practices**
  - *The ability to make strategic decisions when solving problems, to reason, to prove and communicate results*

Fluency

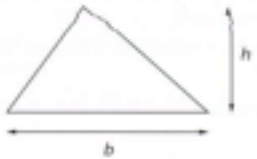
Fluency

Civilisation advances by extending the number of operations which we can perform without thinking about them (Whitehead, 1969)

Facts and skills are should be developed through their subordination and use in rich, significant tasks, rather than through decontextualised memorisation and practice.

Developing fluency: Impoverished tasks

area of triangle = $\frac{1}{2}$ × base × perpendicular height or $A = \frac{1}{2} \times b \times h$



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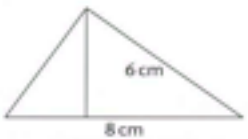
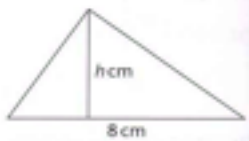


Answer:

a) Area = $\frac{1}{2} \times b \times h = \frac{1}{2} \times 8 \times 6 = 24\,\text{cm}^2$

b) If the area of this triangle is 20 cm², find the perpendicular height of this triangle.



Answer:

b) Area = $\frac{1}{2} \times b \times h$.

$20 = \frac{1}{2} \times 8 \times h = 4h$

$\therefore h = 5\,\text{cm}$

EXERCISE 16.1A

1 Find the areas of these triangles.

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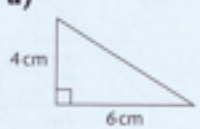

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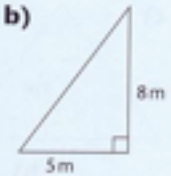

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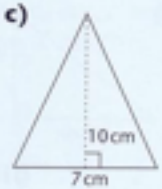

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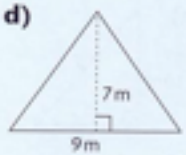

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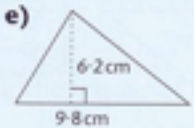

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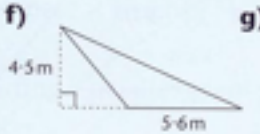

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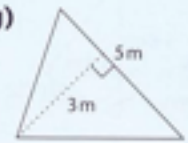

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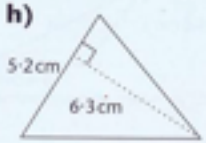

OCR GCSE
Stages 5 & 6
(2007)

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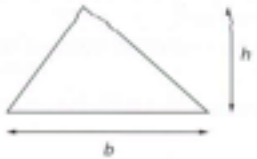

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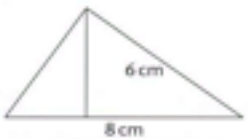

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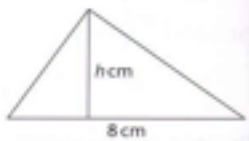

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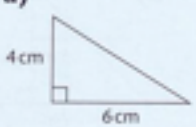

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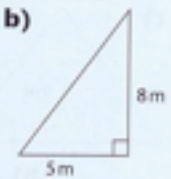

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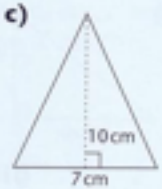

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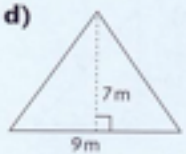

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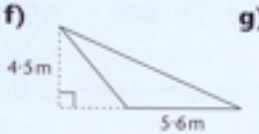

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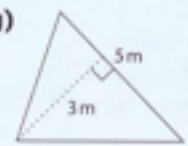

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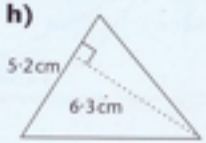

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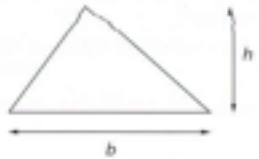

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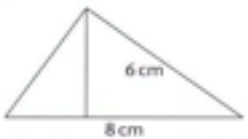

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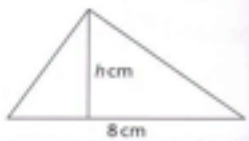

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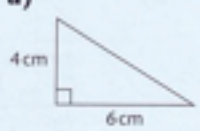

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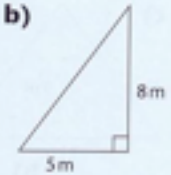

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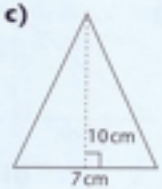

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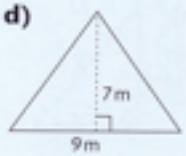

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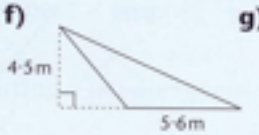

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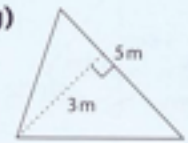

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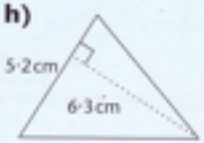

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(2007)

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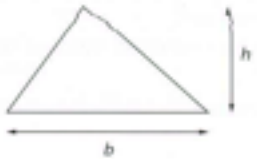

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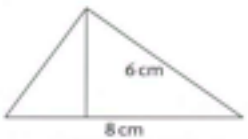

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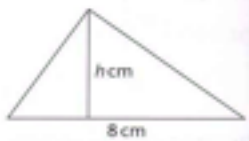

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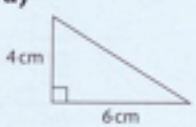

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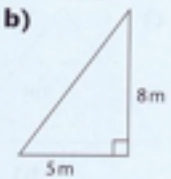

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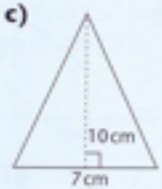

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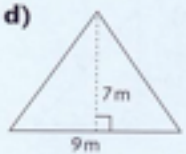

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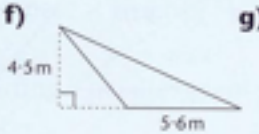

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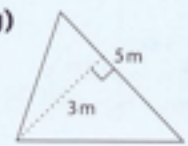

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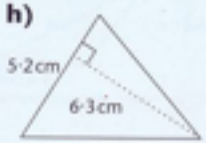


Rule

Example

Exercise

OCR GCSE
Stages 5 & 6
(2007)

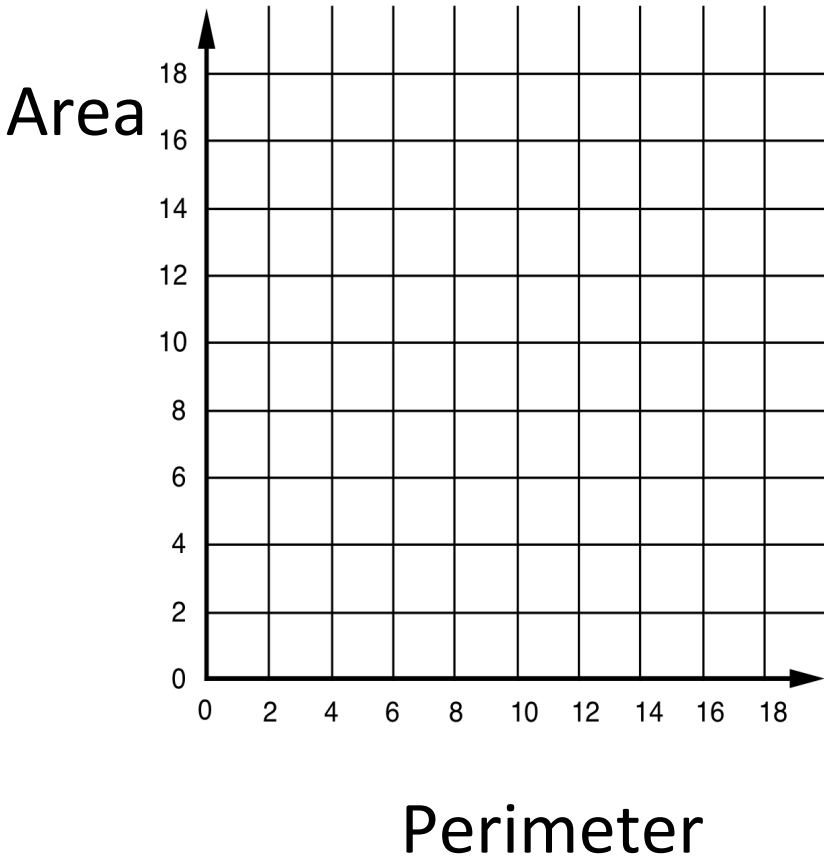
"A common feature of the satisfactory teaching observed was the use of examples followed by practice with many similar questions."

"The teachers typically demonstrated a standard method, giving tips to pupils on how to avoid making mistakes and, sometimes, 'rules' and mnemonics to help them commit the methods to memory."

"This allowed consolidation of a skill or technique but **did not develop problem-solving skills or understanding of concepts**."

(Mathematics: Made to Measure, Ofsted, 2012, p 24)

Developing fluency with rich tasks



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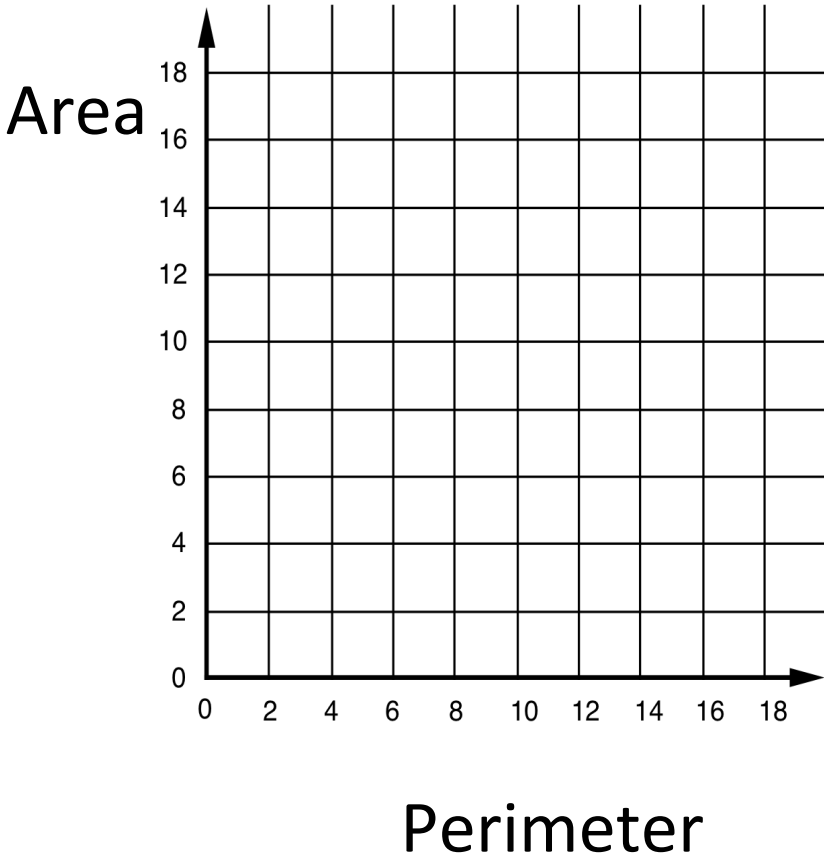

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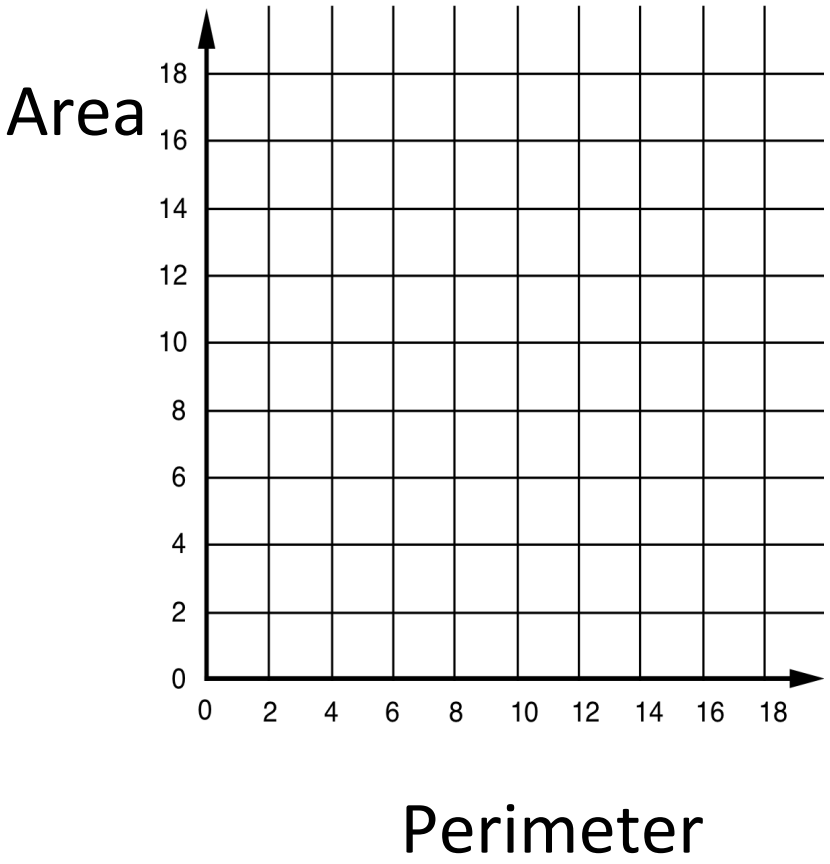

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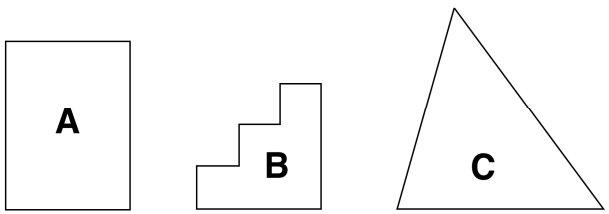

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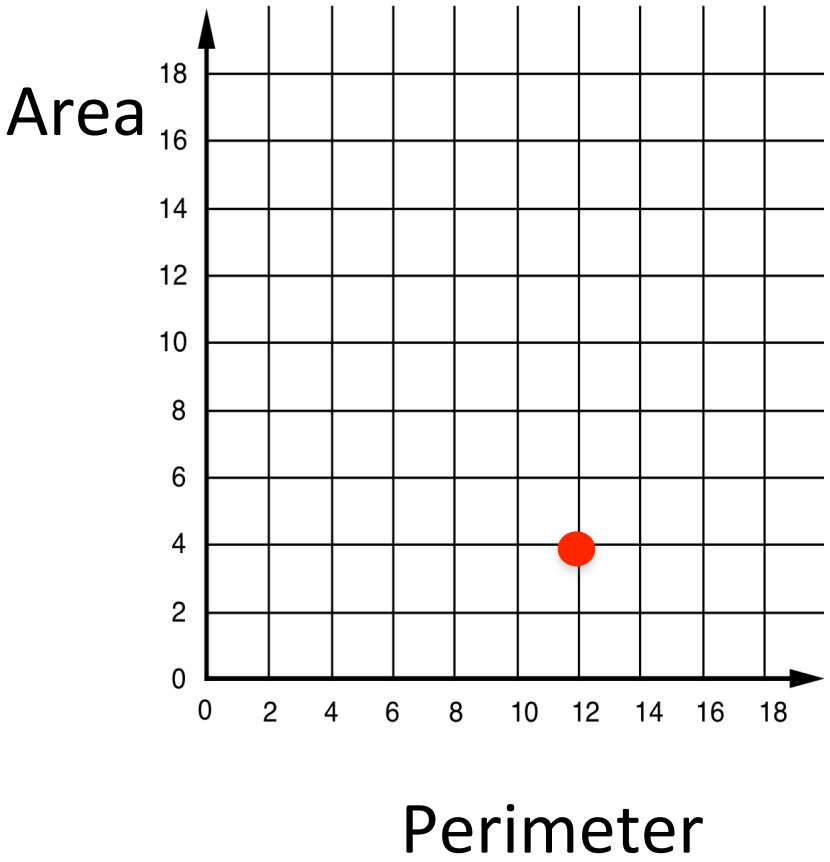

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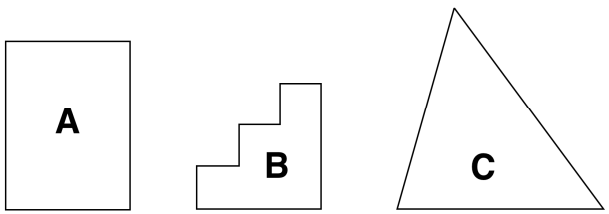
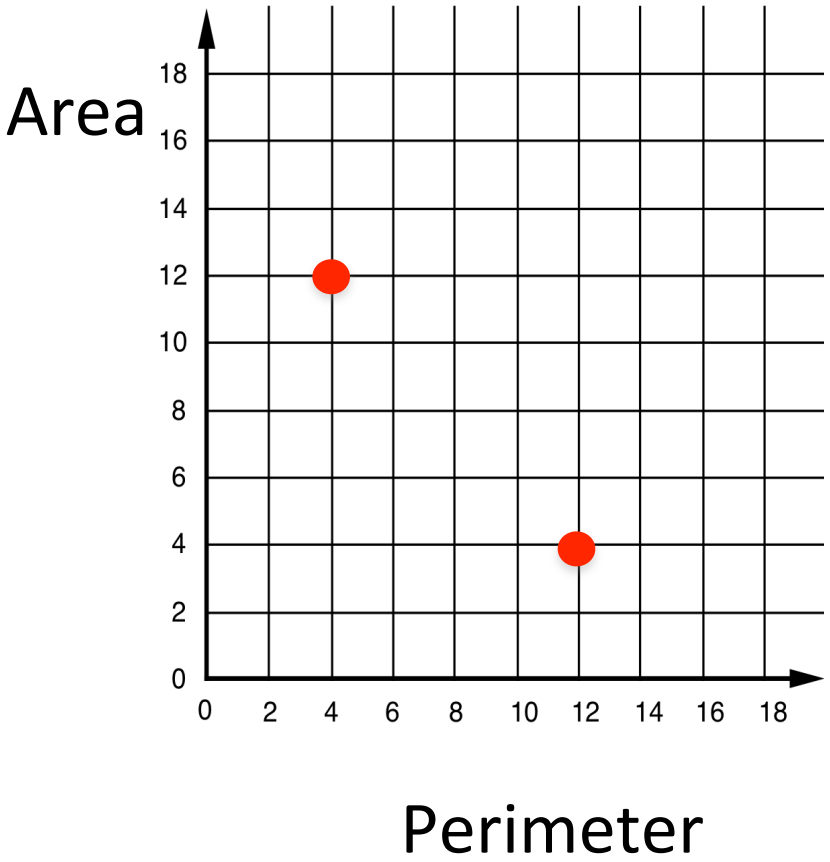
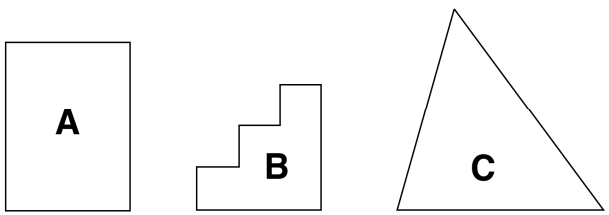


Draw a shape that may be represented by the point (12,4)

Developing fluency with rich tasks



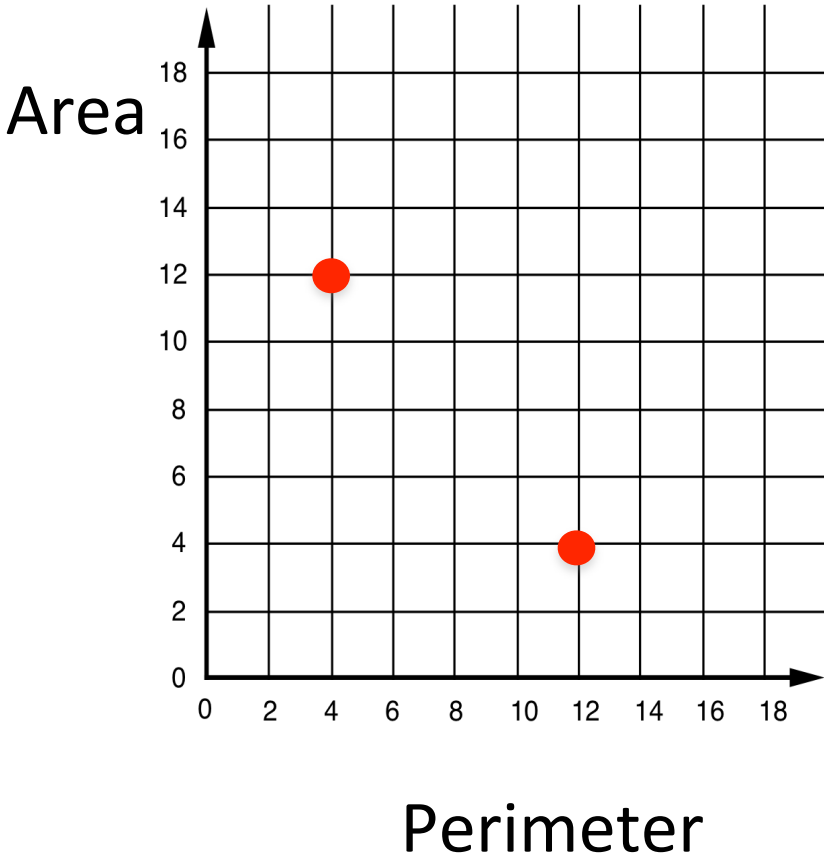
Plot points to represent the following shapes:



Draw a shape that may be represented by the point (12,4)

Draw a shape that may be represented by the point (4,12)

Developing fluency with rich tasks



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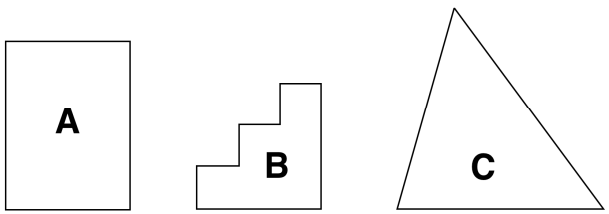
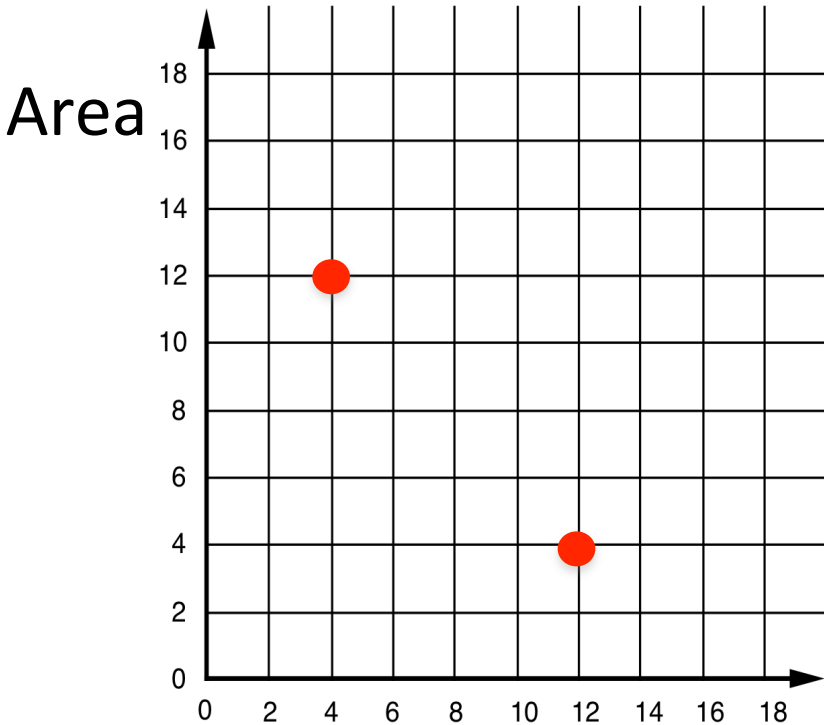


Draw a shape that may be represented by the point (12,4)

Draw a shape that may be represented by the point (4,12)

Which points on the grid represent squares, triangles…..

Developing fluency with rich tasks



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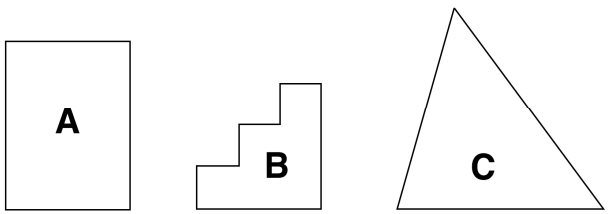


Draw a shape that may be represented by the point (12,4)

Draw a shape that may be represented by the point (4,12)

Which points on the grid represent squares, triangles…..

Which points on the grid represent 'impossible' shapes?

Ofsted (2012) advises that schools should:

- increase the emphasis on **problem solving** across the mathematics curriculum
- develop the expertise of staff:
  - in choosing teaching approaches and activities that foster pupils' deeper **understanding**, including through the use of practical resources, visual images and ICT
  - in checking and probing pupils' understanding during the lesson, and **adapting teaching** accordingly

(Mathematics: Made to Measure, Ofsted, 2012, p 24)

Problem Solving Lessons

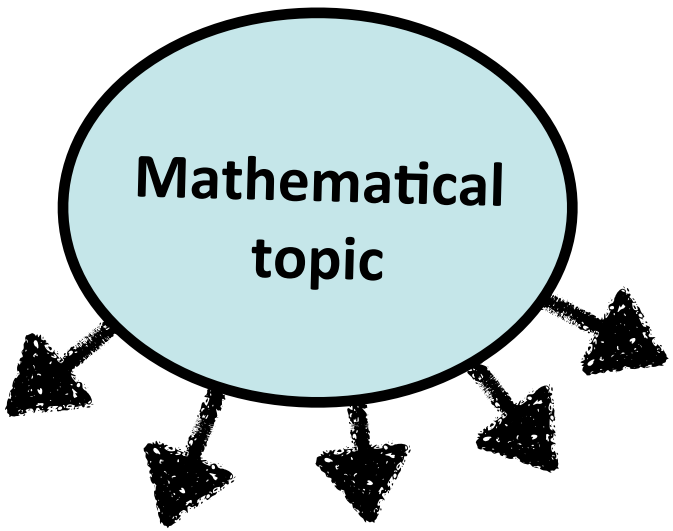
Reveal and develop students' capacity to apply their Math flexibly to non-routine, unstructured problems, both from pure math and from the real world.

- Make sense of problems and persevere in solving them.
- Reason abstractly and quantitatively
- Construct viable arguments and critique the reasoning of others.
- Model with mathematics.
- Use appropriate tools strategically.
- Attend to precision
- Look for and make use of structure
- Look for and express regularity in repeated reasoning.

Understanding Concepts Lessons

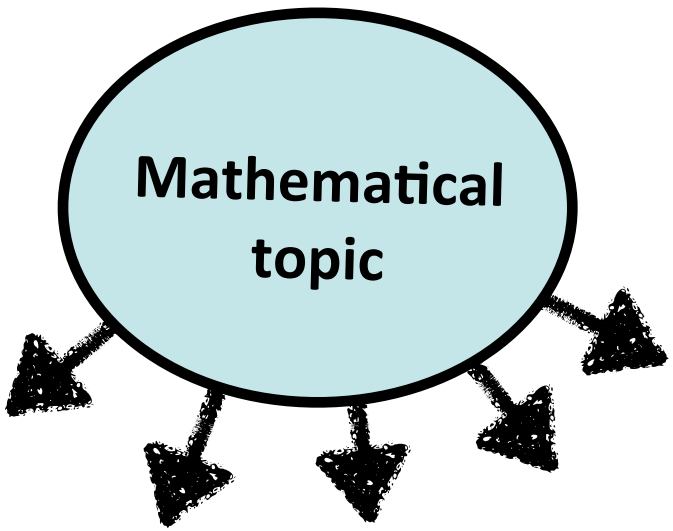
Reveal and develop students' interpretations of significant mathematical ideas and how these connect to their other knowledge.

- Number & Quantity
- Algebra
- Functions
- Modeling
- Statistics and Probability
- Geometry



Exercises and Applications

Concept lessons



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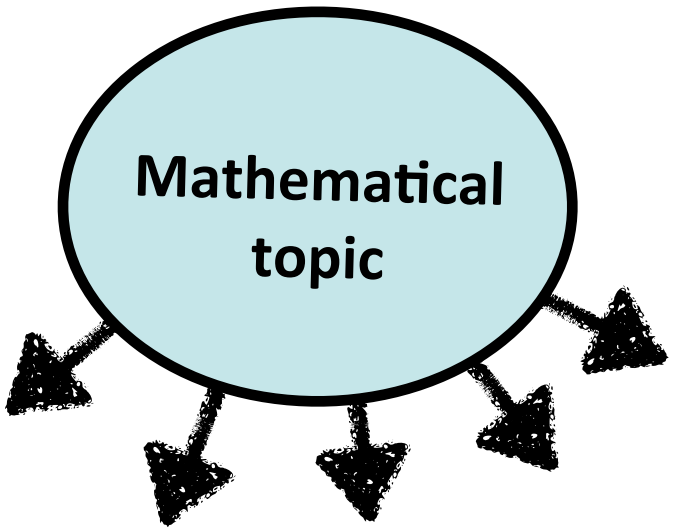
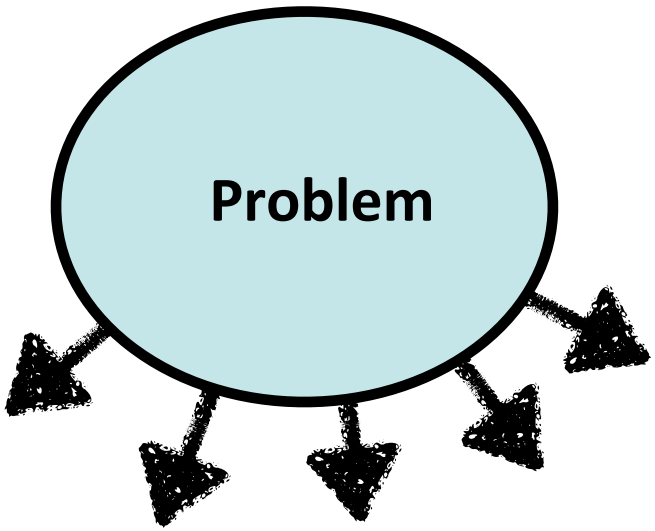


Exercises and Applications

Problem solving lessons



Choose appropriate mathematical tools

Formative assessment - Adaptive teaching

Students and teachers

Using evidence of learning

To adapt teaching and learning

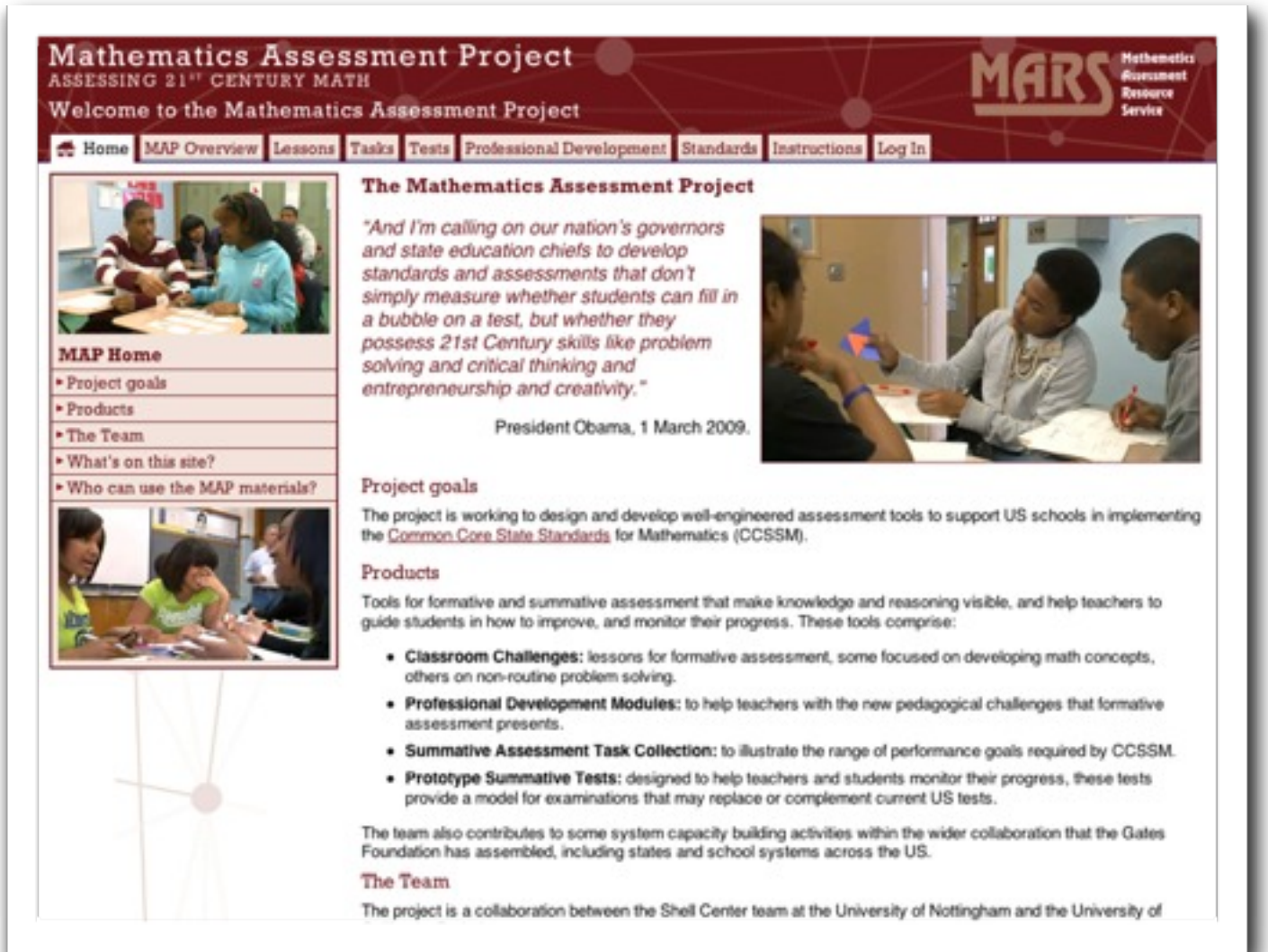
To meet immediate needs

Minute-to-minute and day-by-day

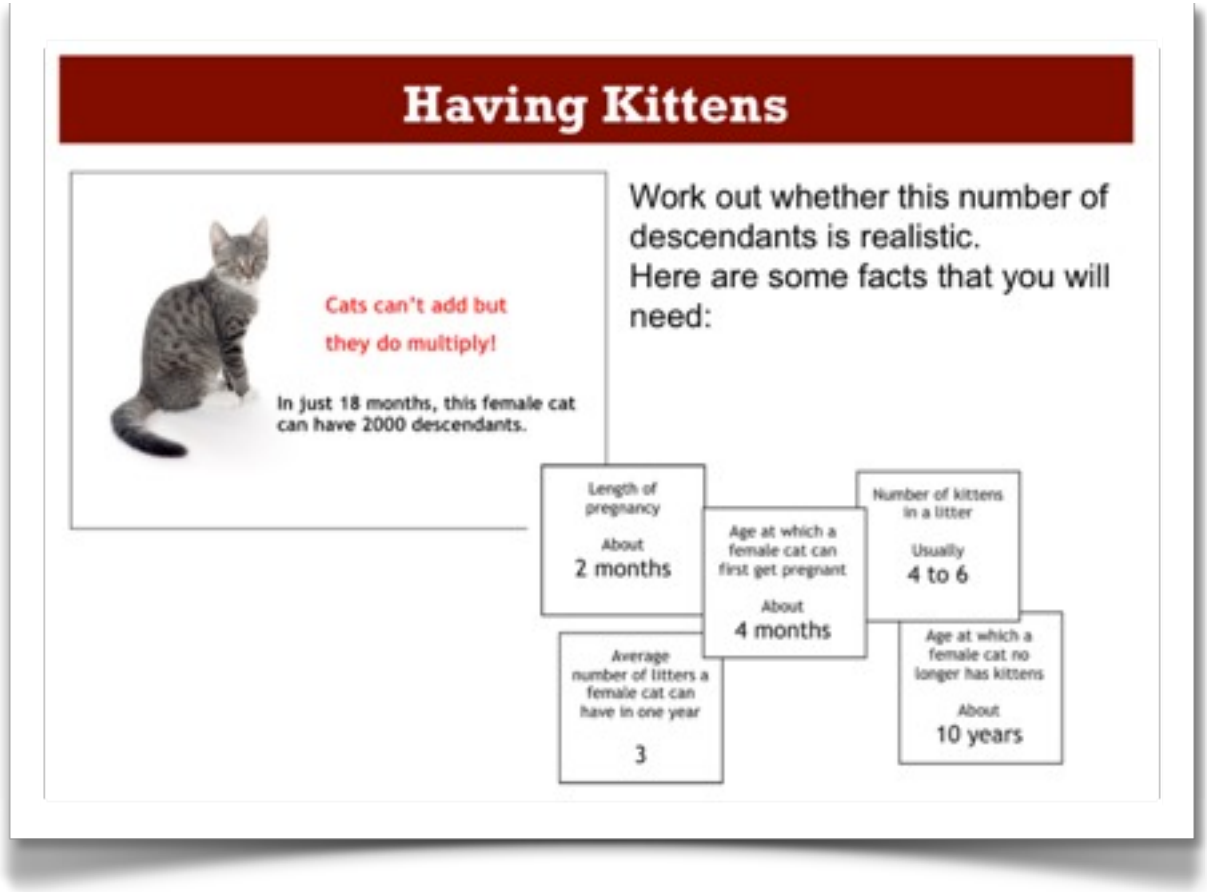
(Thompson and Wiliam, 2007)

Mathematics Assessment Project (MAP)

http://map.mathshell.org/materials/



100 Classroom Challenges + PD Support
Grades 6, 7, 8 and High School



Detailed lesson plans

SUGGESTED LESSON OUTLINE

Introduction: *Cats and Kittens* (10 minutes)

Begin the lesson by briefly reintroducing the problem:

Do you remember the problem I asked you to do last time? I have had a look at your work and I have some questions on it.

Today you are going to work together to improve your initial attempts.

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Use your mini-whiteboard to note answers to these questions.

It is helpful to ask pupils to write their ideas a mini-whiteboard using a felt-tipped pen, as you will be able to monitor their work more easily. This will also help pupils to share their ideas easily later in the lesson.

Collaborative activity: Producing a joint solution (20 minutes)

The time given for this activity is an approximate guide. You may find your class would benefit from spending more (or less) time on this activity.

Organize the class into small groups of two or three students and give each group a large sheet of paper and a felt-tipped pen. Ask students to try the task again, this time combining their ideas.

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Suggested structure for lesson

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Suggested instructions and questions

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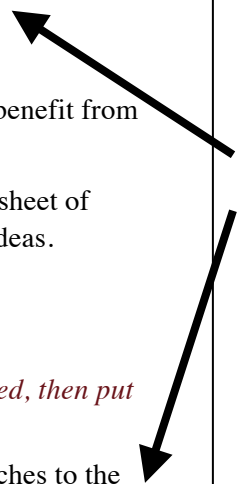

Suggested teaching approaches

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Formative lessons in Problem Solving

Key Processes (UK) / Modeling (US)

- **Represent / Formulate**

Identify the maths in situations, and select methods, representations and tools to use

- **Analyse / Solve**

Analyse and reason while working on problems

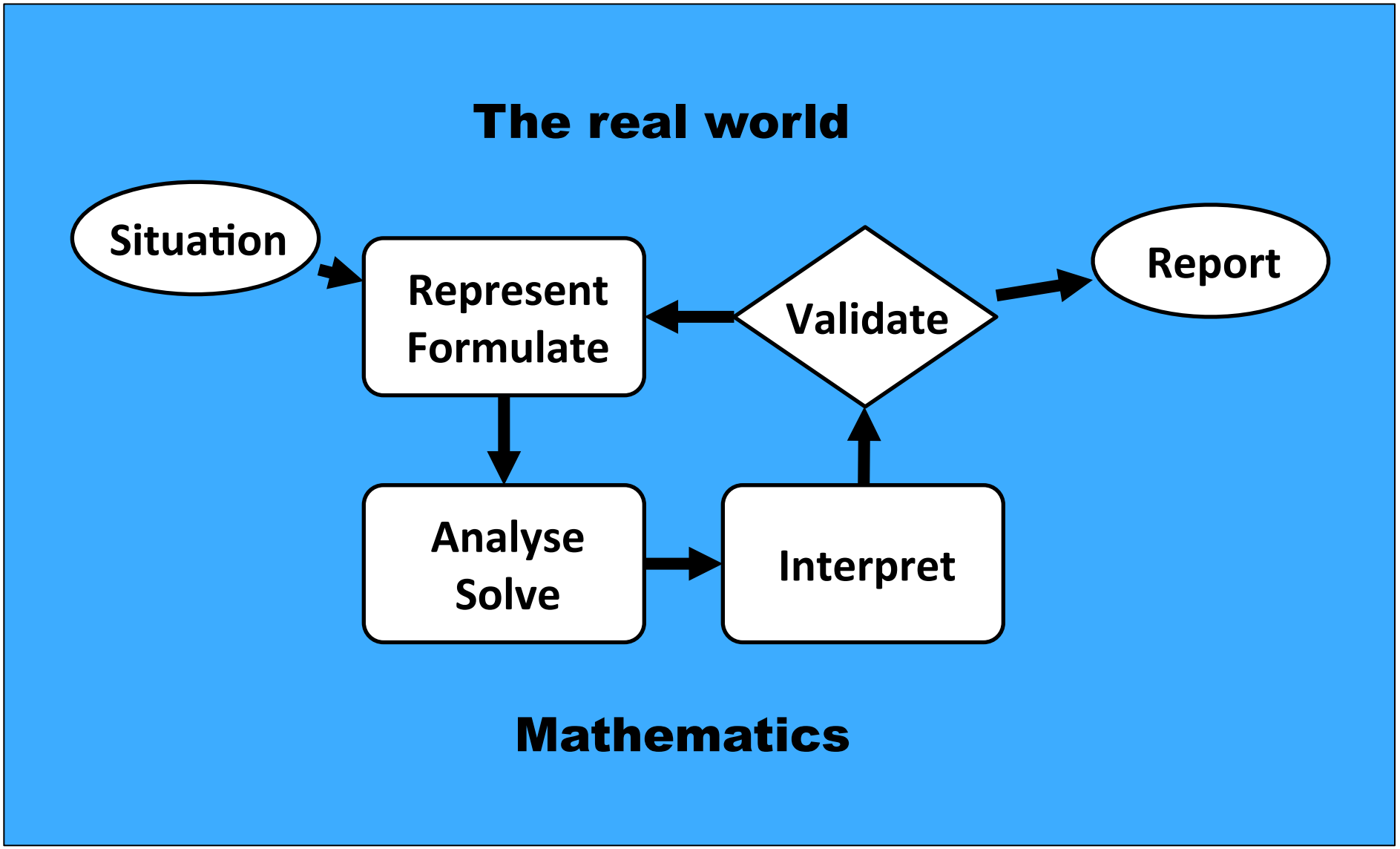
- **Interpret and evaluate / Interpret and validate**

Interpreting findings and strategies and constructing arguments

- **Communicate and reflect / Report**

Share findings and thinking effectively.

What is modelling?

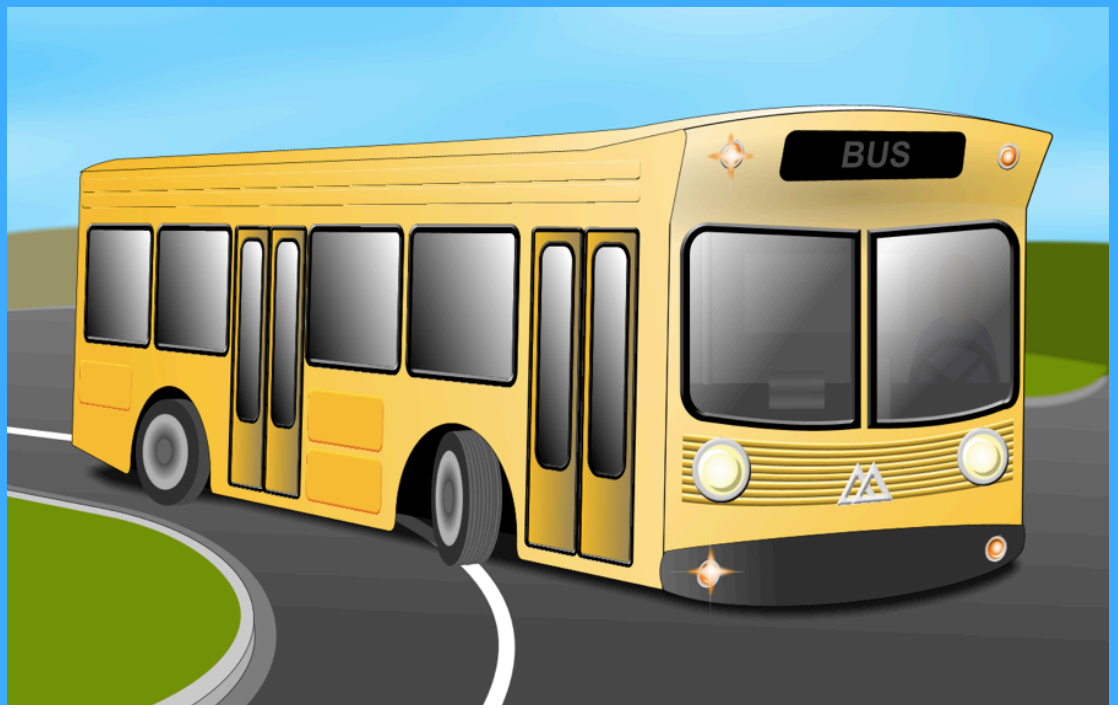


The modelling process

Situation

The modelling process

Situation



How much do buses cut corners?

The modelling process

Situation → Formulate

The modelling process

Situation → Formulate

Make simplifying assumptions

- suppose each wheel turns in a circular arc.
- just consider how the wheels cut the corner.

The modelling process

Situation → Formulate

List and select important variables

- distance between rear and front wheels (w);
- radius of bend (r);
- amount rear wheel cuts the corner (x)

The modelling process

Situation → Formulate

List and select important relationships

- If I increase w, what is the effect on x?
- If I increase r, what is the effect on x?
- can we sketch graphs?

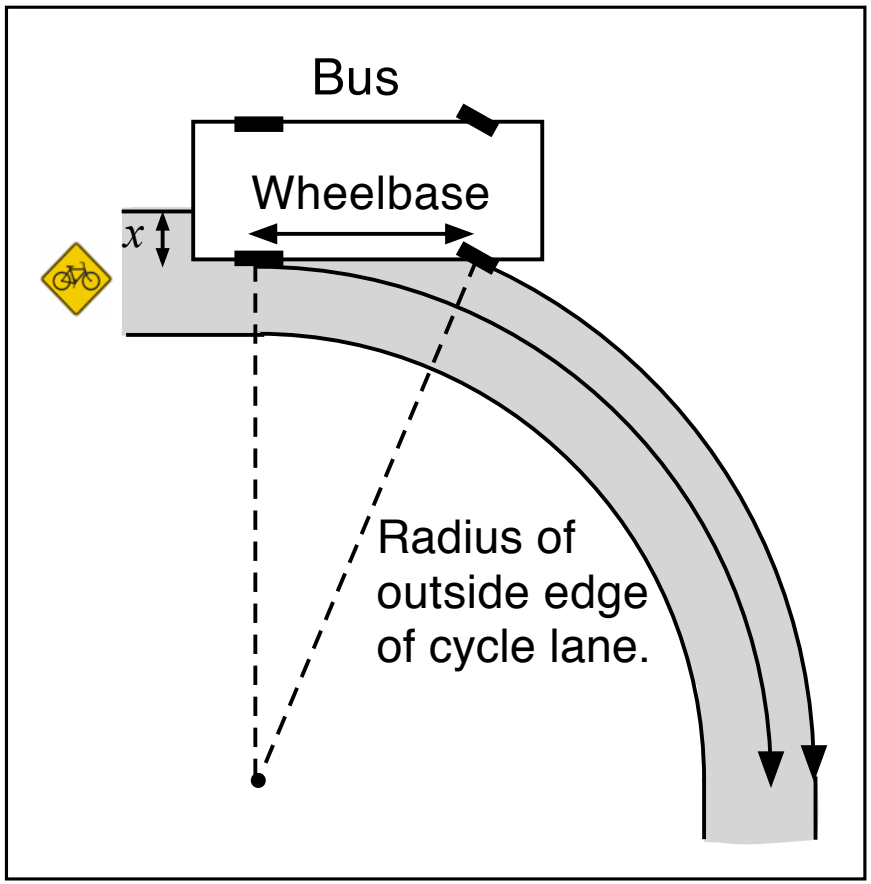
The modelling process

Situation

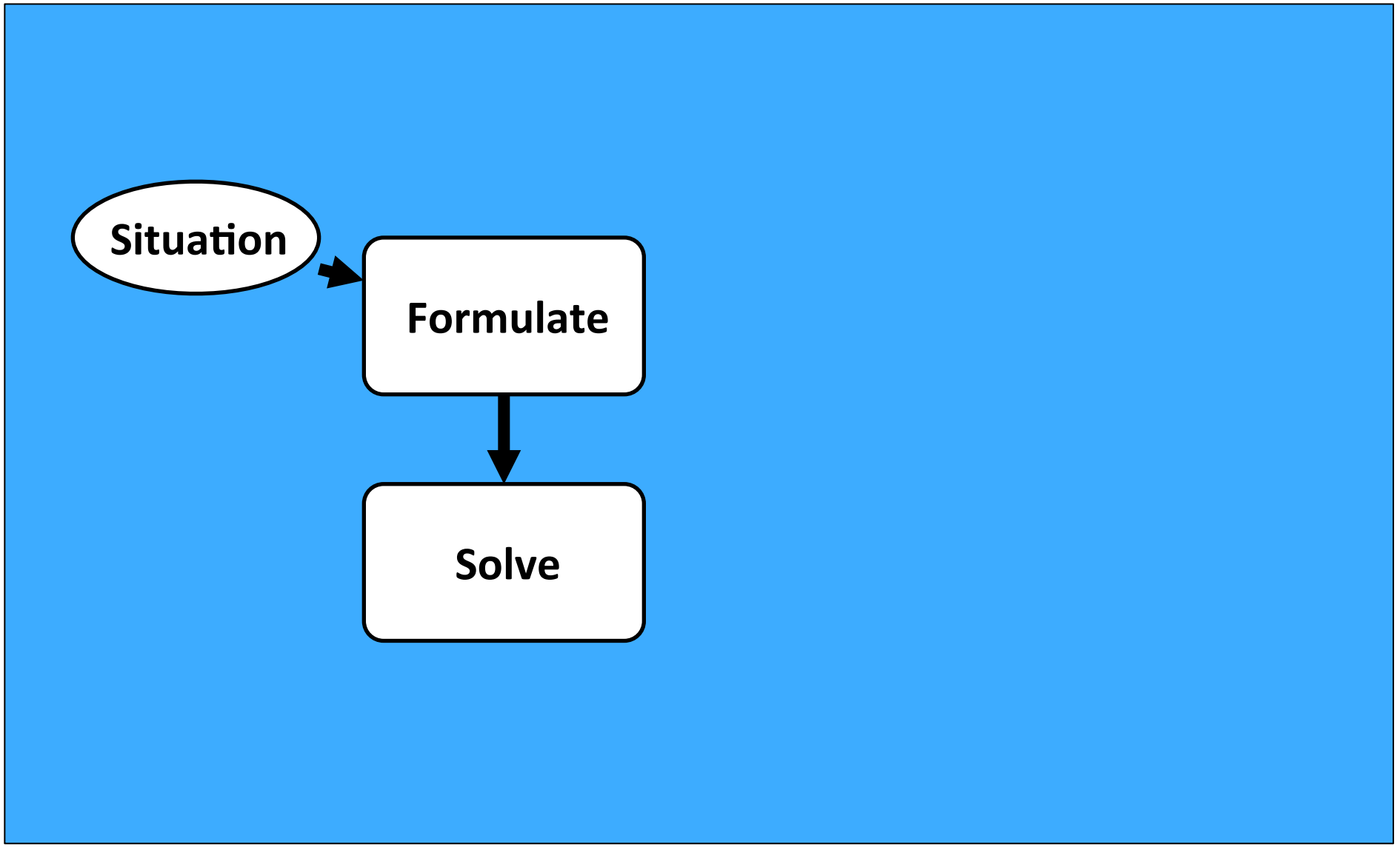
Formulate

Choose helpful representations

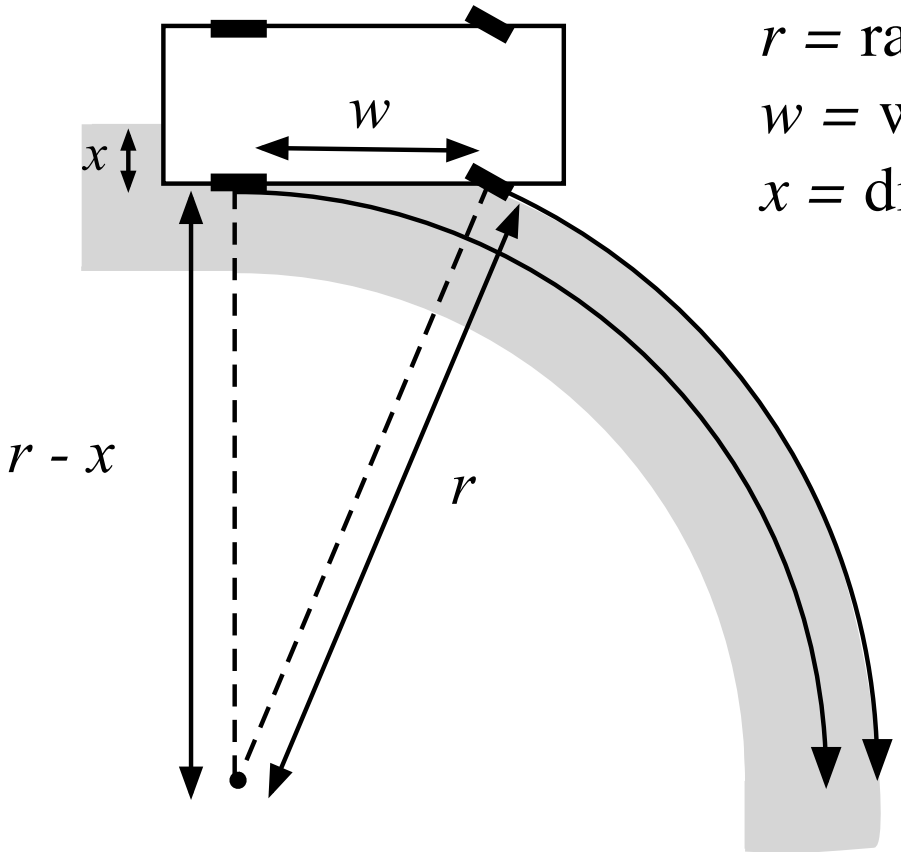
(physical model, diagrams, graphs)



The modelling process

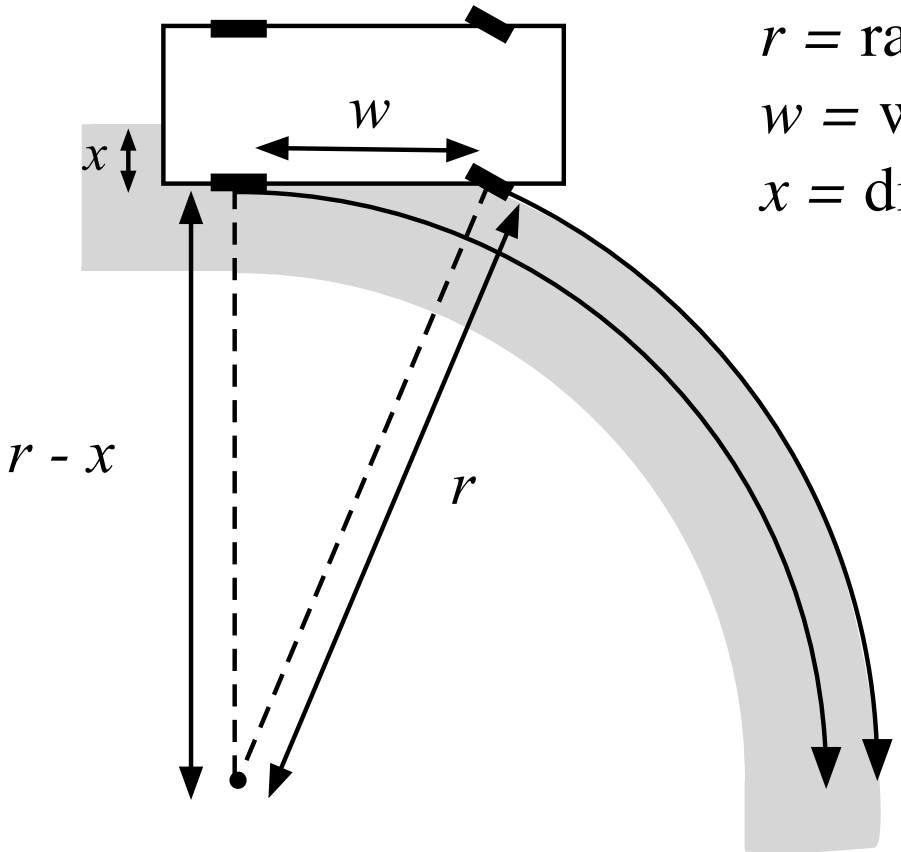


Solve



r = radius of outside edge of cycle lane
w = wheelbase of bus
x = distance bus cuts into the cycle lane

Solve



r = radius of outside edge of cycle lane
w = wheelbase of bus
x = distance bus cuts into the cycle lane

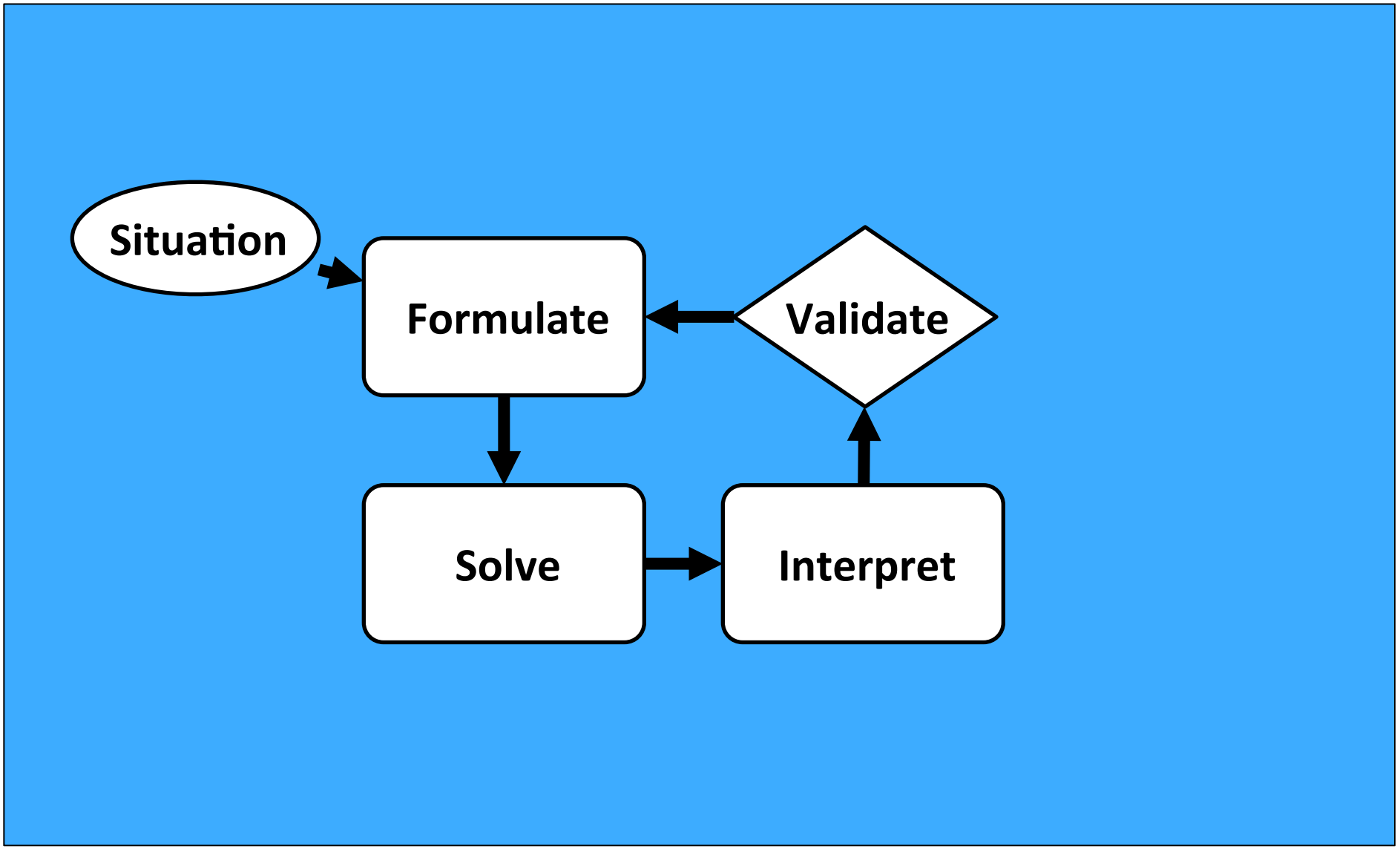
$$(r-x)^2 + w^2 = r^2$$

$$\Rightarrow r^2 - 2rx + x^2 + w^2 = r^2$$

$$\Rightarrow x^2 - 2rx + w^2 = 0$$

$$\Rightarrow x = r \pm \sqrt{r^2 - w^2}$$

The modelling process



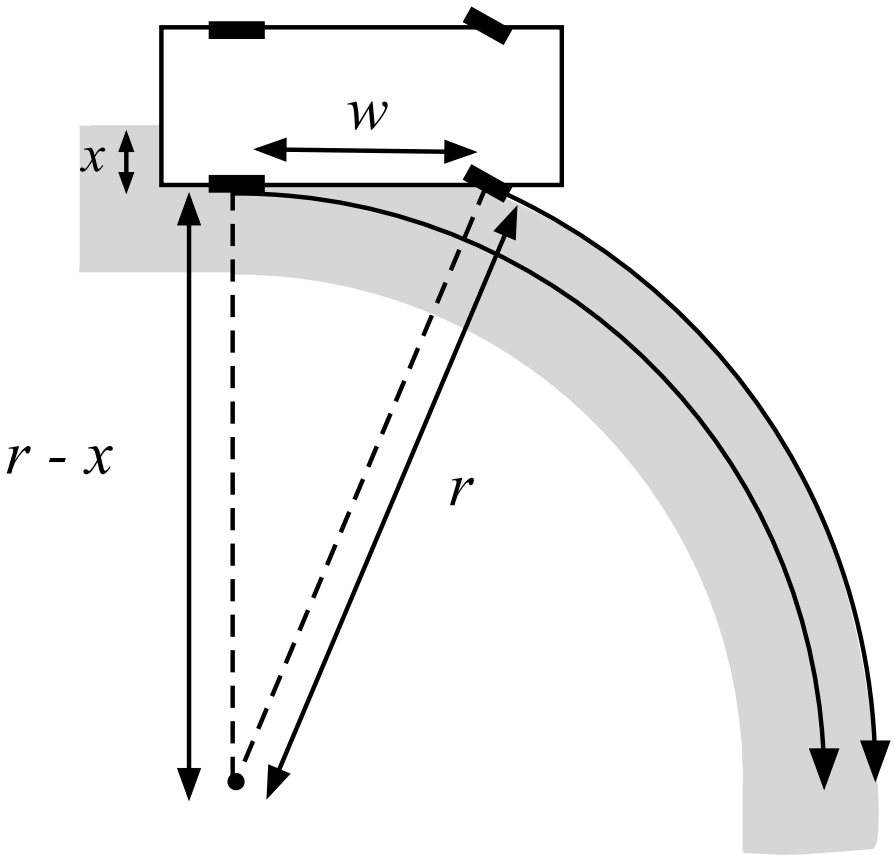
Interpret and validate

Try a special case.
Make a prediction and check it.
Suppose that:
$w = 4$ metres and
$r = 5$ metres.

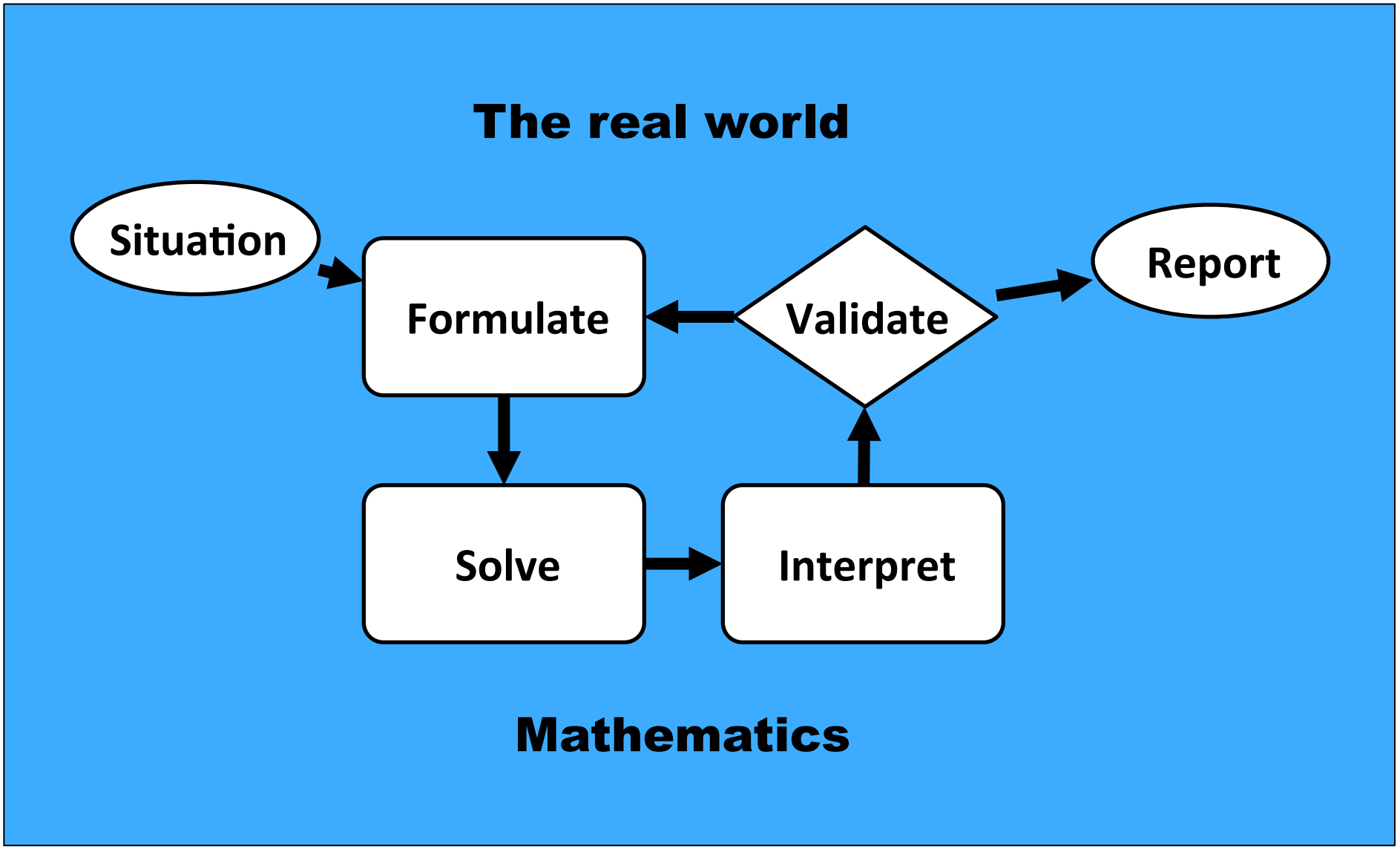
$$x = 5 \pm \sqrt{25 - 16}$$

$$x = 8 \text{ or } 2$$

What does the $x = 8$ solution mean?
Is this reasonable?



The modelling process



What types of problems? What student roles?

- **Plan and organise**
  - Find an optimum solution subject to constraints.
- **Design and make**
  - Design an artefact or procedure and test it
- **Model and explain**
  - Invent, explain models, make reasoned estimates
- **Explore and discover**
  - Find relationships, make predictions
- **Interpret and translate**
  - Deduce information, translate representations
- **Evaluate and improve**
  - An argument, a plan, an artefact

Problem Solving Assessment Lesson

- **Initial, individual, unscaffolded problem**
  - Students tackle the problem unaided.
    Teacher assesses work and prepares qualitative feedback.
- **Individual work**
  - Students write responses to teacher's feedback
- **Collaborative work**
  - Students work together to produce a joint solution
- **Students compare different approaches**
  - Students discuss student work in small groups, then as a whole class
- **Individual work**
  - Students improve their solutions to the initial problem,
    or one very much like it.

Problem Solving Assessment Lesson

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Cats and Kittens

Cats can't add but they can multiply!

In just 18 months, this female cat can have 2000 descendants

Make sure your cat cannot have kittens

Length of pregnancy
About
2 months

Age at which a female cat can first get pregnant
About
4 months

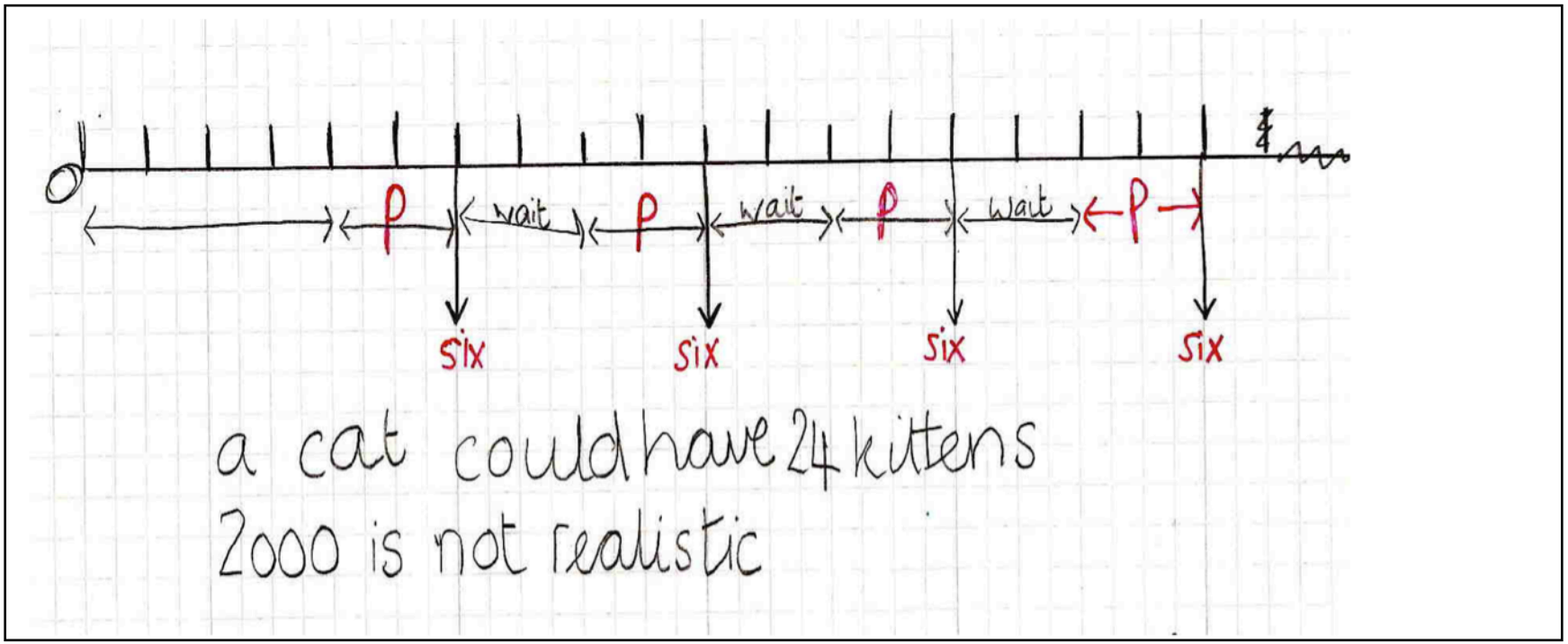
Number of kittens in a litter
Usually
4 to 6

Age at which a female cat no longer has kittens
About
10 years

Average number of litters a female cat can have in one year
3

Is this figure of 2000 realistic ?

Sample student work



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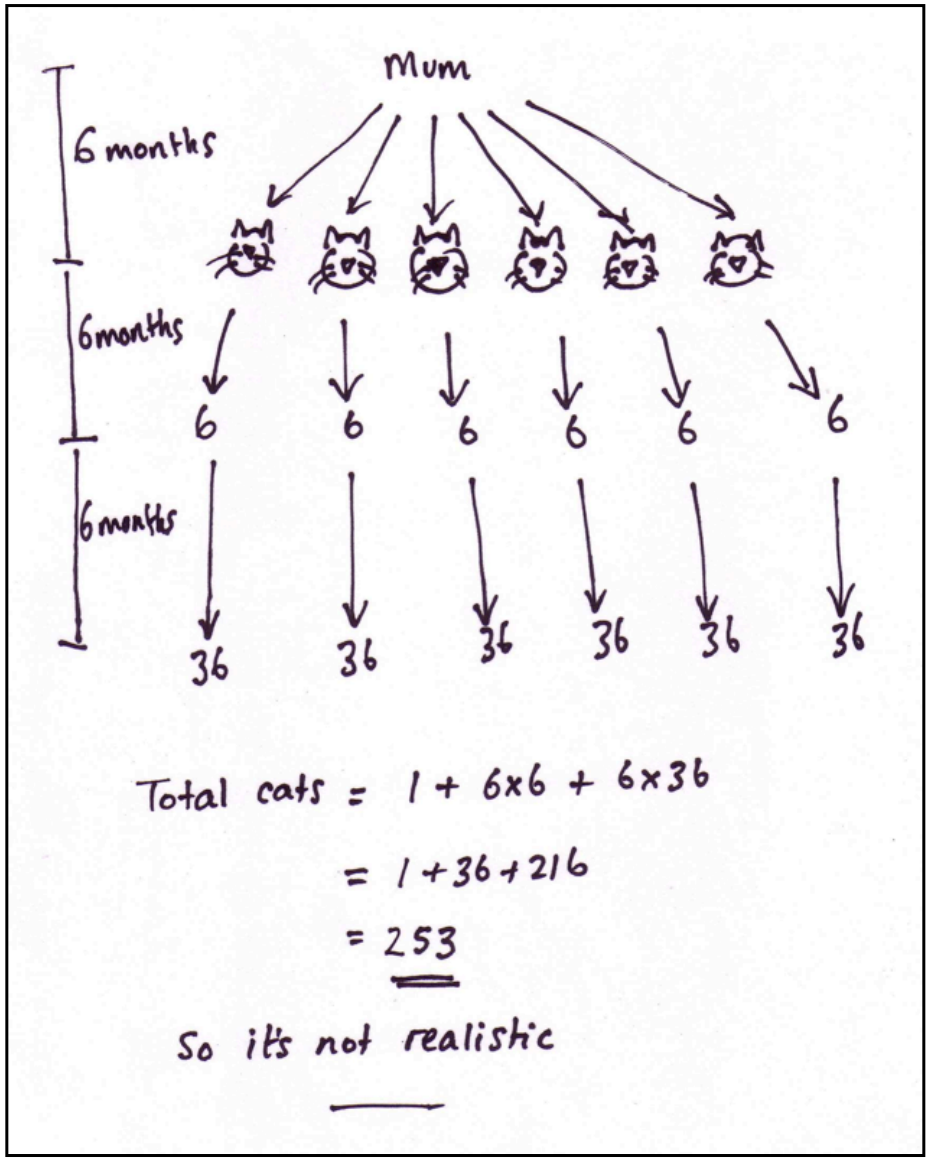

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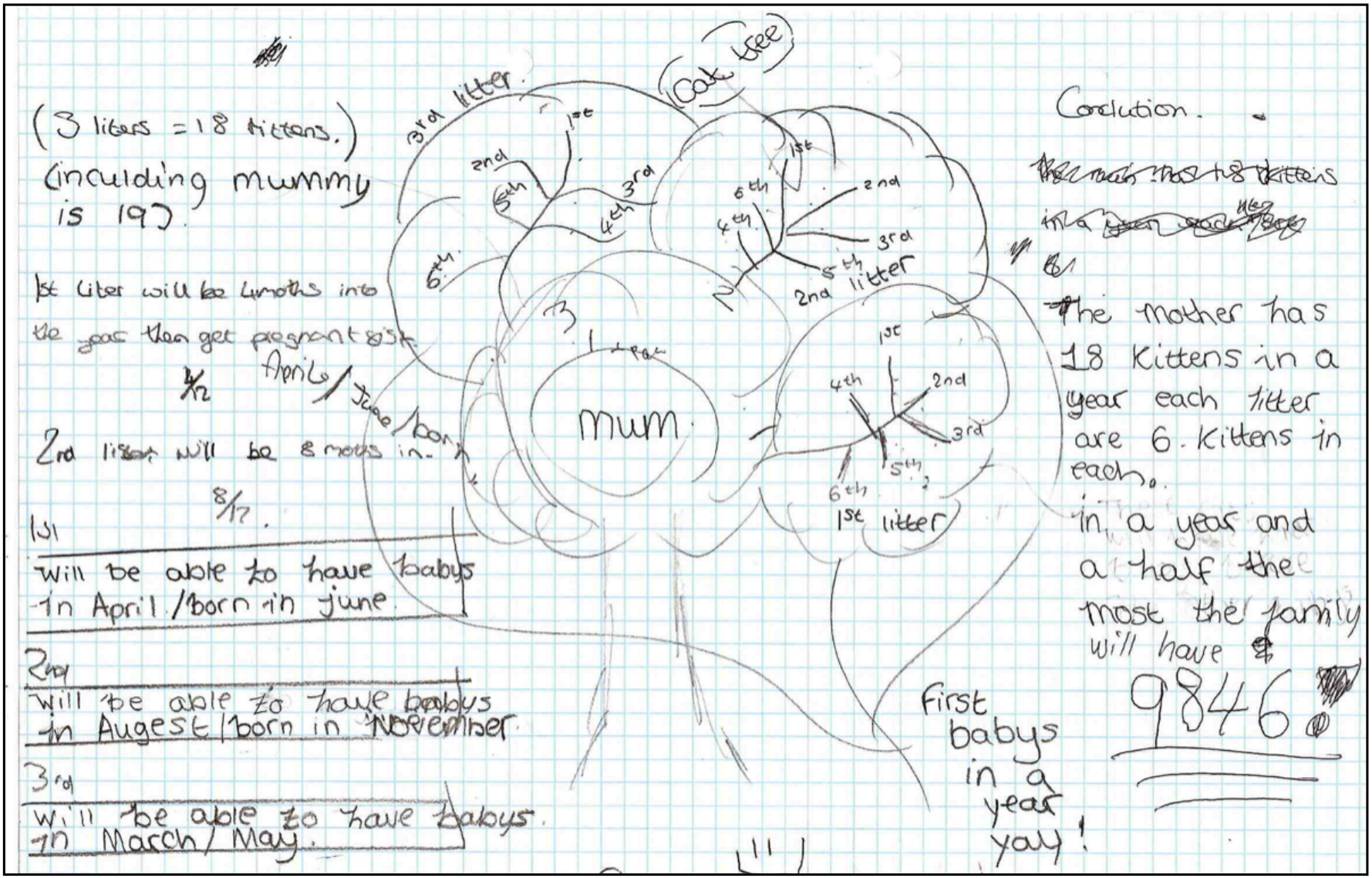

Common issues tables

Has difficulty starting	Can you describe what happens during first five months?
Does not develop suitable representation	Can you make a diagram or table to show what is happening?
Work is unsystematic	Could you start by just looking at the litters from the first cat? What would you do after that?
Develops a partial model	Do you think the first litter of kittens will have time to grow and have litters of their own? What about their kittens?
Does not make clear or reasonable assumptions	What assumptions have you made? Are all your kittens are born at the beginning of the year?
Makes a successful attempt	How could you check this using a different method?

Problem Solving Assessment Lesson

Problem Solving Assessment Lesson

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  - Students tackle the problem unaided.
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Problem Solving Assessment Lesson

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  - Students tackle the problem unaided.
    Teacher assesses work and prepares qualitative feedback.
- **Individual work**
  - Students write responses to teacher's feedback

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  - Students discuss student work in small groups, then as a whole class

Maths Around
the World

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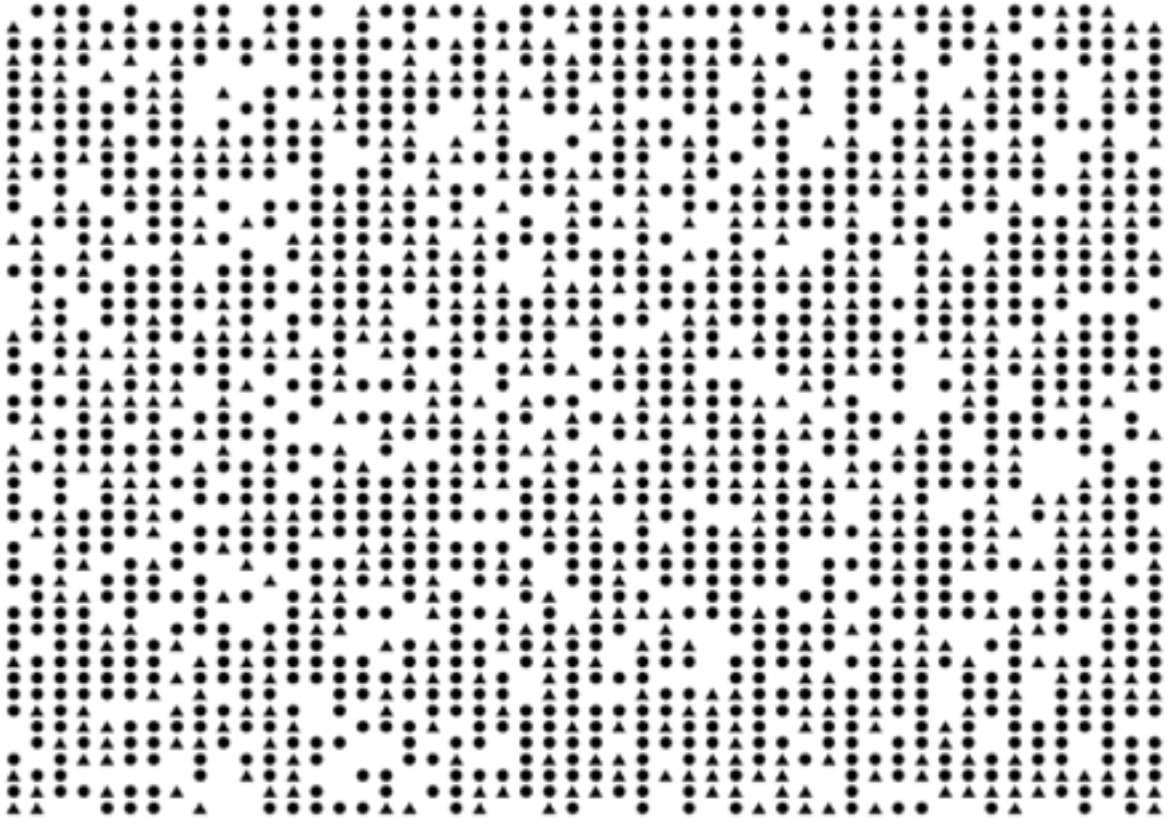
- **Collaborative work**
  - Students work together to produce and share joint solutions
- **Students compare different approaches using sample work**
  - Students discuss student work in small groups, then as a whole class
- **Whole class discussion: the payoff of mathematics**
  - Students improve their solutions to the initial problem,
    or one very much like it.
- **Individual reflection**
  - Students write about what they have learned.

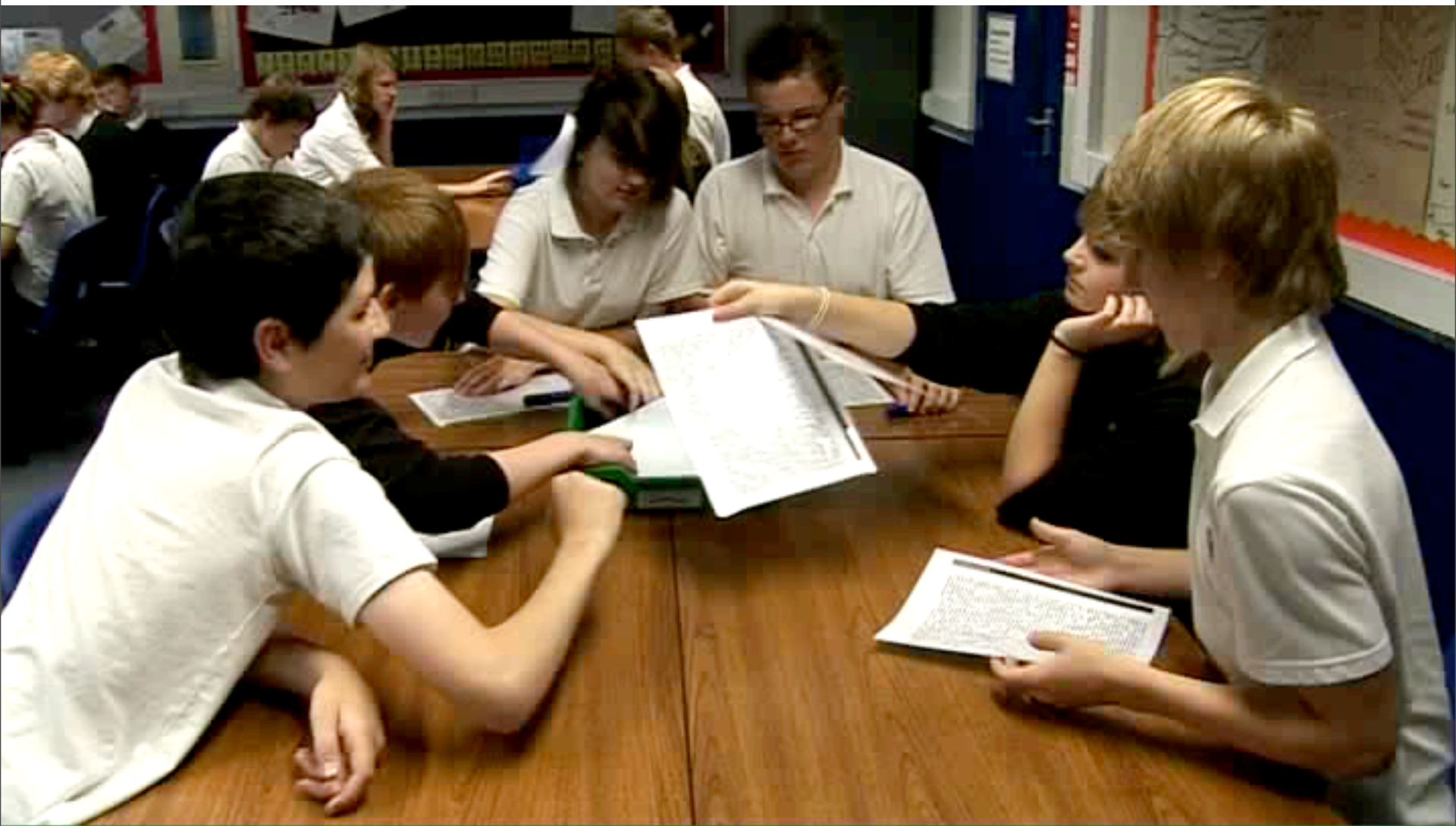
Counting Trees



● The circles show **old** trees

▲ The triangles show **young** trees

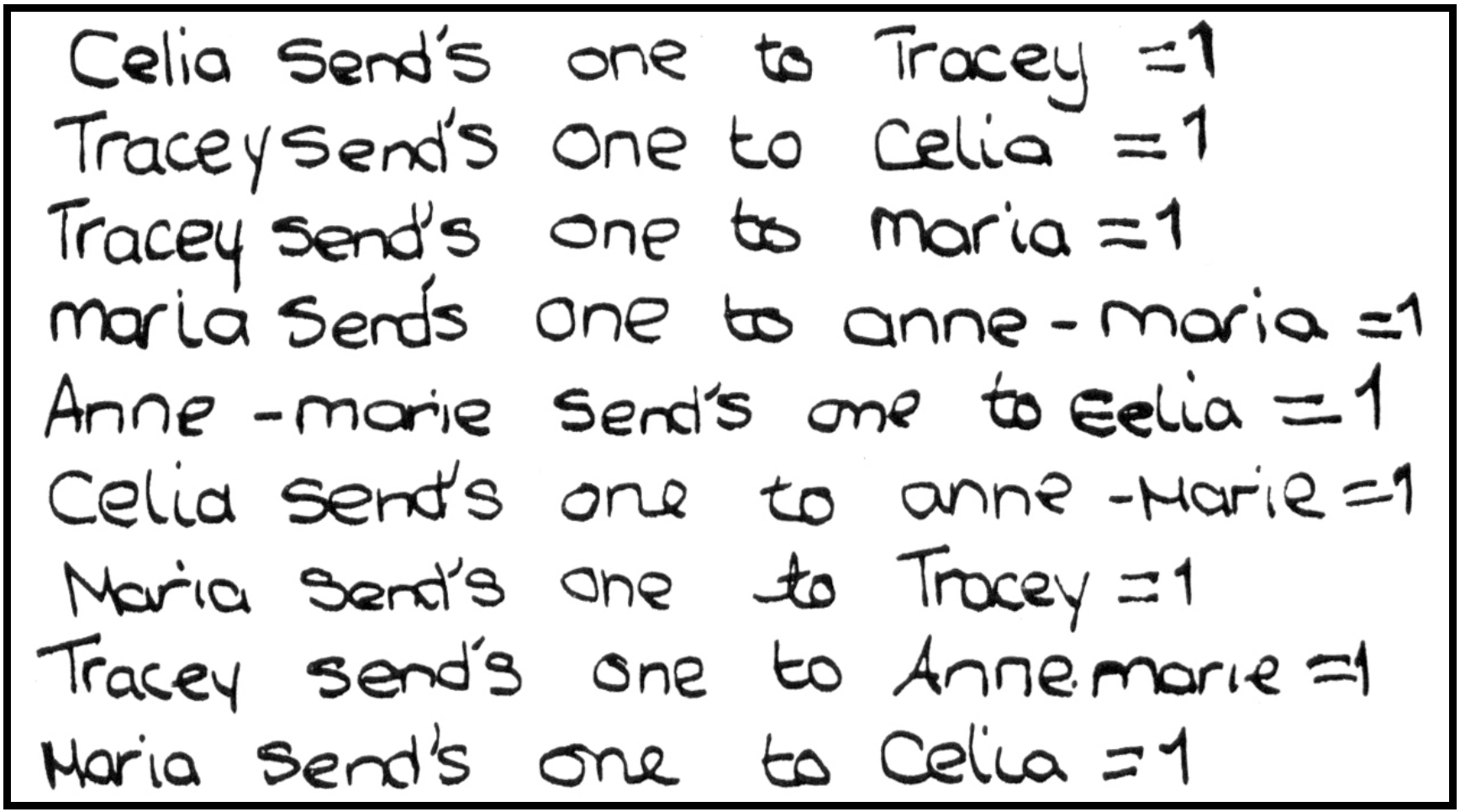
- Think of a method you could use to estimate the number of trees of each type.
- Explain the method fully
- Use your method to estimate the number of old trees and young trees



Text Messaging

- How many text messages are sent if four people all send messages to each other?
- How many text messages are sent with different numbers of people?
- Approximately how many text messages would travel in cyberspace if everyone in your school took part?
- Can you think of other situations that would give rise to the same mathematical relationship?

Sample work stimulates a variety of approaches



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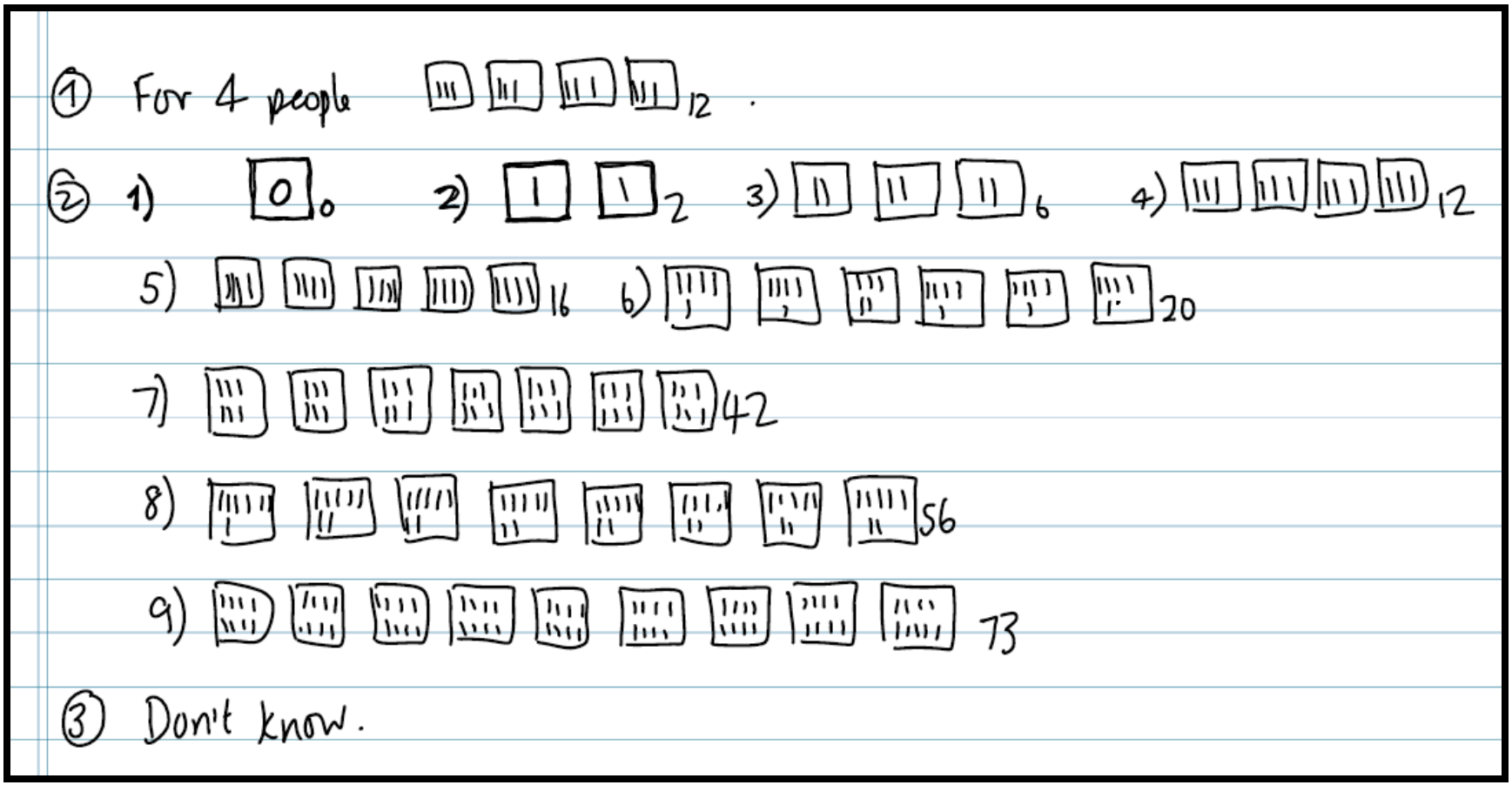


Sample work stimulates a variety of approaches

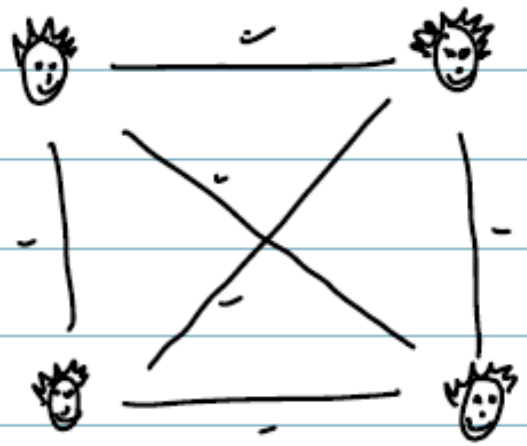
	Amy	Belinda	Suzie	Mary	Tom
Amy	—	Text	Text	Text	Text
Belinda	Text	—	Text	Text	Text
Suzie	Text	Text	—	Text	Text
Mary	Text	Text	Text	—	Text
Tom	Text	Text	Text	Text	—

= 12 texts for 4 people

Tom adds 8 more texts = 20 altogether.

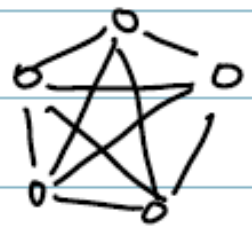
For more people you add extra rows and columns.

Sample work stimulates a variety of approaches



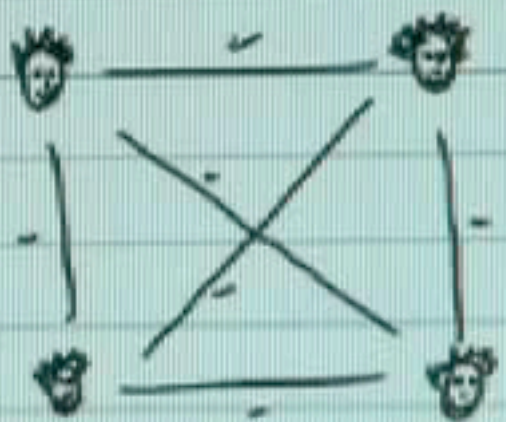
= 6 texts

3



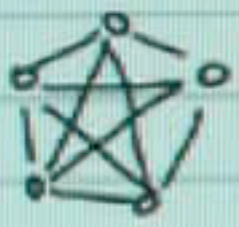
People	1	2	3	4	5
texts	0	1	3	6	10

Chris's answer



= 6 texts

3



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Where is the assessment?

- Teachers gather information on what students can do unaided;
- Teachers offer support to students, as this is needed;
- Students get constructive feedback via other students, and the teacher, as student work is discussed;
- Students act on feedback by improving their responses;
- Teachers get feedback on learning by comparing performance before and after.

Formative lessons for developing concepts

Principles for Concept Lessons

- **Expose and explore students' existing ideas**
  - "pull back the rug"
- **Confront with implications, contradictions, obstacles**
  - provoke 'tension' and 'cognitive conflict'
- **Resolve conflict through discussion**
  - allow time for formulation of new concepts.
- **Generalise, extend and link learning**
  - applying to new contexts.

Concept Development Cycle

- **Initial, individual task**
  - An assessment task is presented and a range of responses are evoked. The task is put to one side.
- **Collaborative discussion**
  - Prior conceptions are discussed and debated. Teacher aims to provoke cognitive conflict through questioning.
- **Whole class discussion**
  - Pre-conceptions are explicitly challenged.
- **Revisit initial task or a similar one**
  - The assessment task is re-examined and responses are improved. Students describe what they have learned.

Task "genres" that generate discussion

- **Classifying, naming and defining objects**
  - what is the same and what is different?
- **Interpreting multiple representations**
  - what is another way of showing this?
- **Analysing and testing generalisations**
  - "always, sometimes or never true?"
- **Modifying problems & exploring structure**
  - what happens if I change this?
  - How will it affect this?

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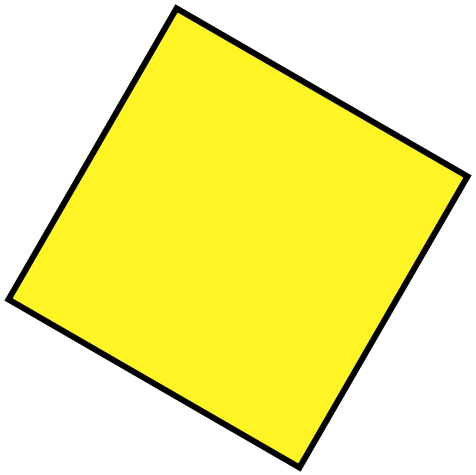
Classifying and defining

Four equal sides

Diagonals meet at right angles



Two pairs of parallel sides

4 lines of symmetry

Two equal diagonals

Rotational symmetry of order 4

Four right angles

Classifying and defining

Classifying and defining

$$y = -(x - 4)(x + 2)$$

Classifying and defining

Quadratic function

$$y = -(x-4)(x+2)$$

Maximum value of 9

Crosses x axis at (4,0)

Factorises with integers

Crosses y axis at (0,8)

Line of symmetry $x = 1$

Classifying and defining

Quadratic function

Maximum value of 9

$$y = -(x-4)(x+2)$$

Crosses x axis at (4,0)

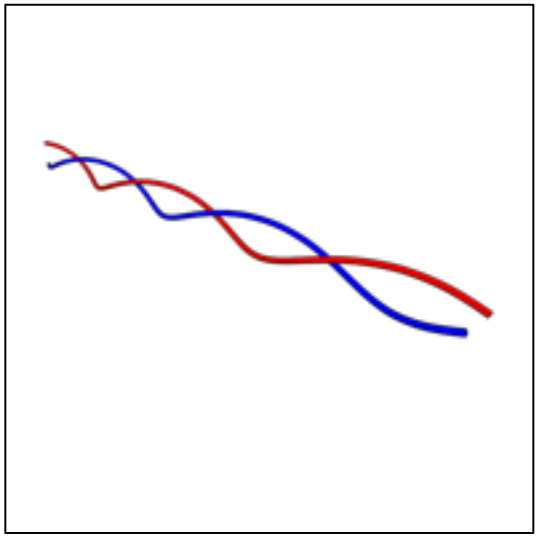
Factorises with integers

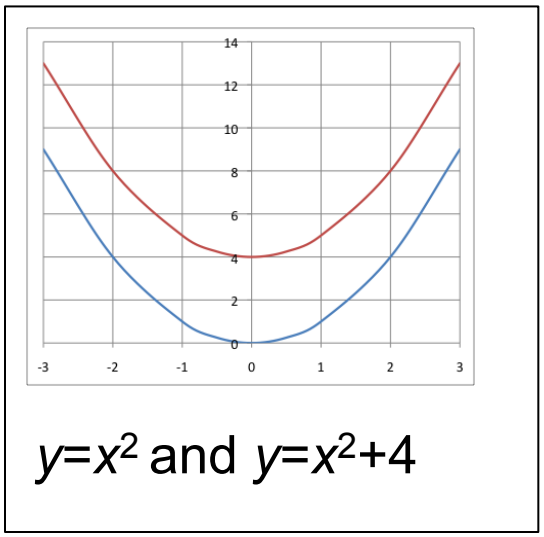
Crosses y axis at (0,8)

Line of symmetry $x = 1$

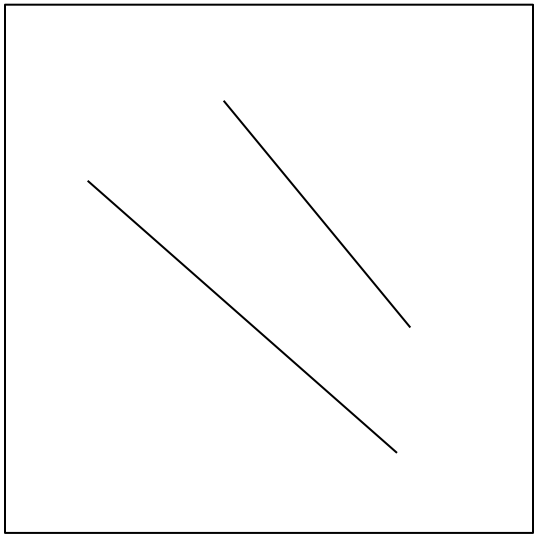
Which *pairs* of statements *define* the quadratic?

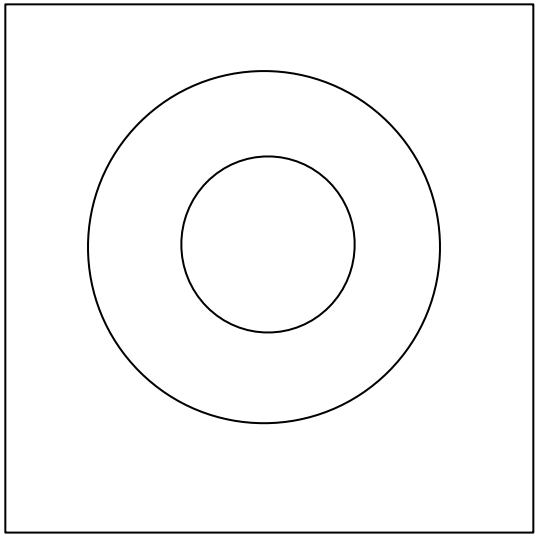
Defining carefully ... Parallel?



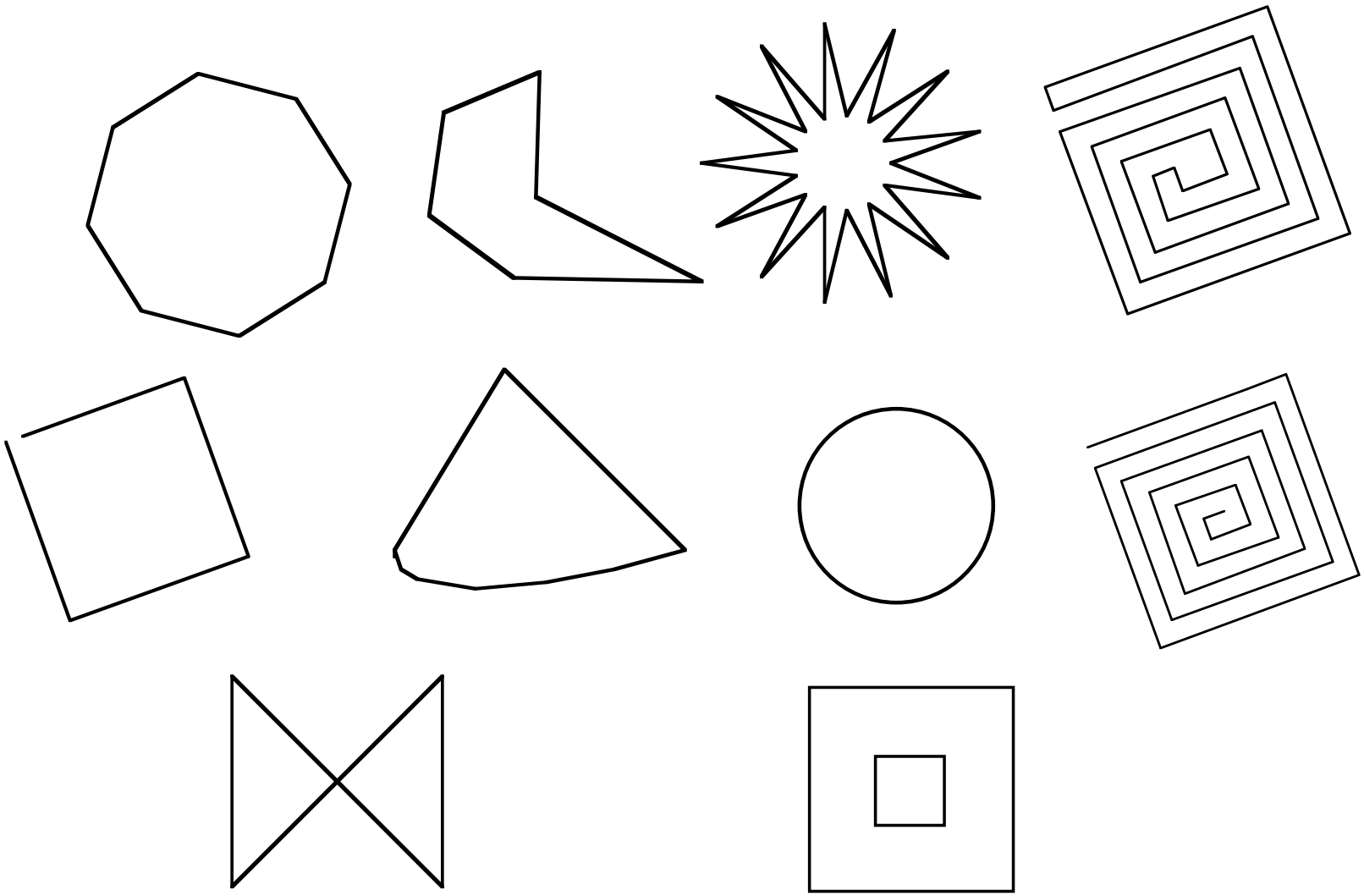


$y=x^2$ and $y=x^2+4$

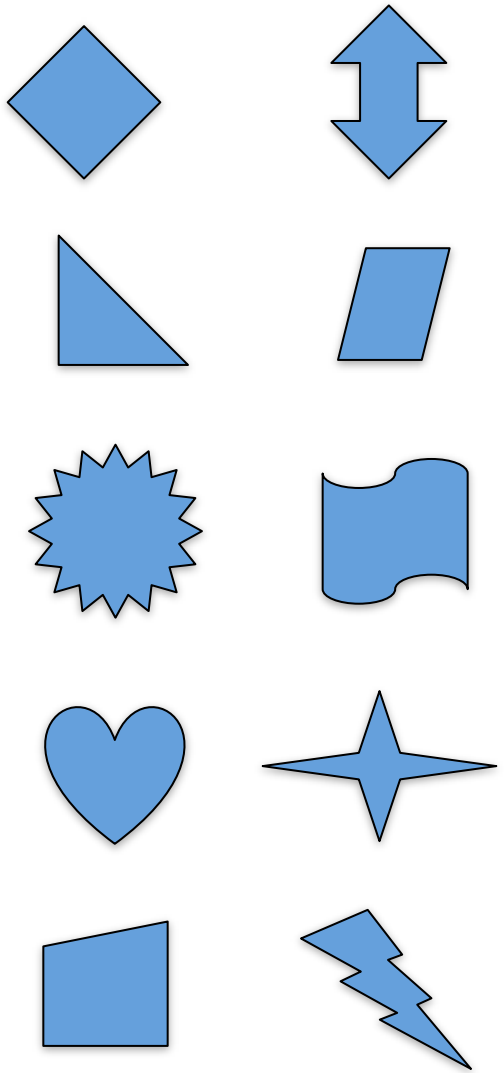




Defining carefully... Polygon?



	No rotational symmetry	Rotational symmetry
No lines of symmetry		
One or two lines of symmetry		
More than two lines of symmetry		



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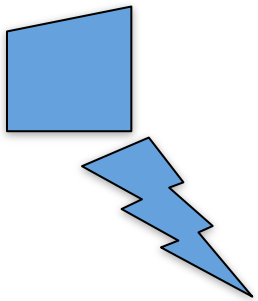
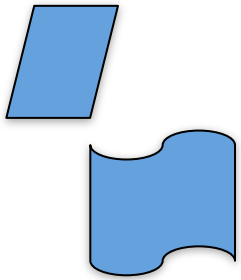
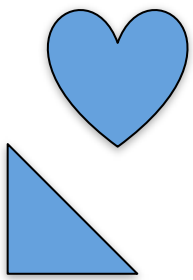
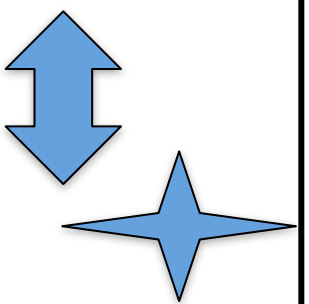
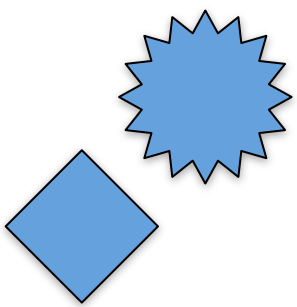
No lines of symmetry		
One or two lines of symmetry		
More than two lines of symmetry		

Is it possible to find a shape that has no rotational symmetry which has more than two lines of symmetry?

$y = x^2 + 2x + 4$

$y = 4x^2 - 4x + 1$

$y = x^2 - 5x + 4$

$y = x^2 - 4x + 4$

$y = x^2 + 7x - 3$

$y = 2x^2 - 5x - 3$

	Factorizes with integers	Does not factorize with integers
Two *x* intercepts		
One *x* intercept		
No *x* intercepts		

Is it possible to find a quadratic function $y=f(x)$ that factorizes but has no x intercepts?

	Factorizes with integers	Does not factorize with integers
Two x intercepts	$y = x^2 - 5x + 4$	$y = 2x^2 - 5x - 3$ $y = x^2 + 7x - 3$
One x intercept	$y = x^2 - 4x + 4$	$y = 4x^2 - 4x + 1$
No x intercepts		$y = x^2 + 2x + 4$

Classifying functions

	Odd	Even	Neither
One to one			
Many to one			

$f(x) = x + 3$

$f(x) = \ln x$

$f(x) = x^4 - 4x$

$f(x) = \sin x$

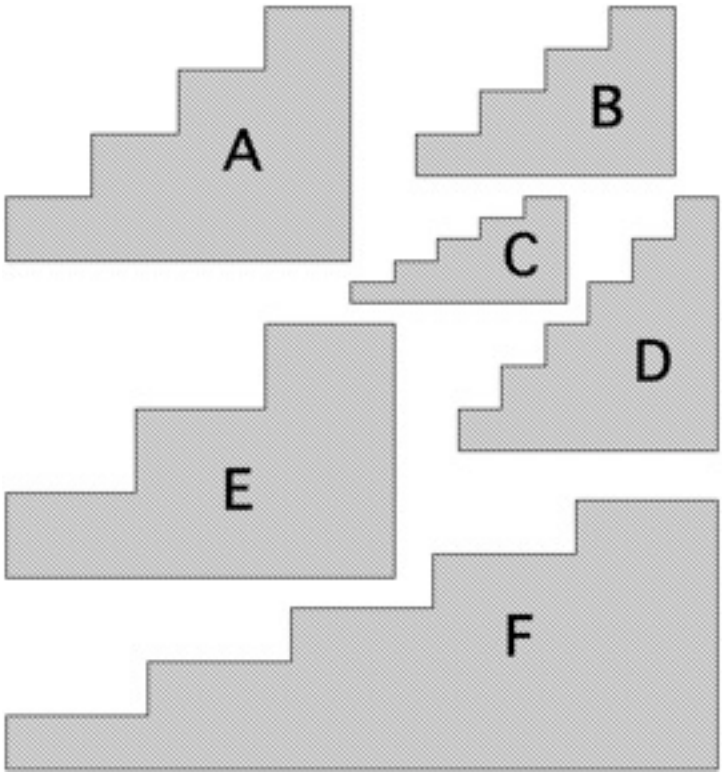
$f(x) = \cos 2x$

$f(x) = x^2 - x + 6$

Ground rules for students

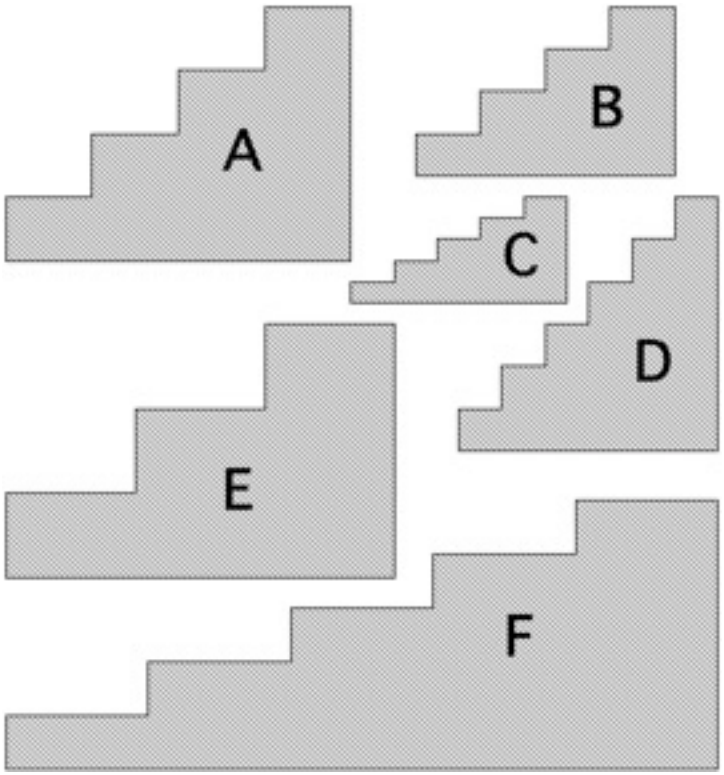
- Give everyone a chance to speak.
- Listen without interrupting.
- Check that everyone else listens.
- Try to understand what is said.
- Build on what others have said.
- Challenge what is said.
- Demand good explanations.
- Treat opinions with respect.
- Share responsibility.
- Try to reach agreement.

Challenging definitions: “Steepness”



Challenging definitions: “Steepness”

Put these staircases in order of steepness.



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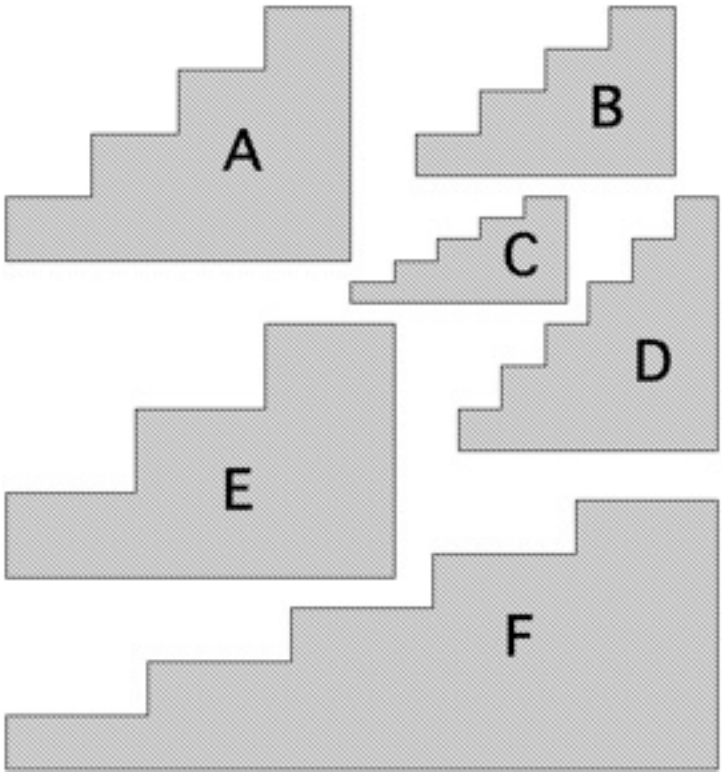
Put these staircases in order of steepness.

Is...

“Height of step - length of step” a good measure of steepness?



Challenging definitions: “Steepness”

Put these staircases in order of steepness.

Is...

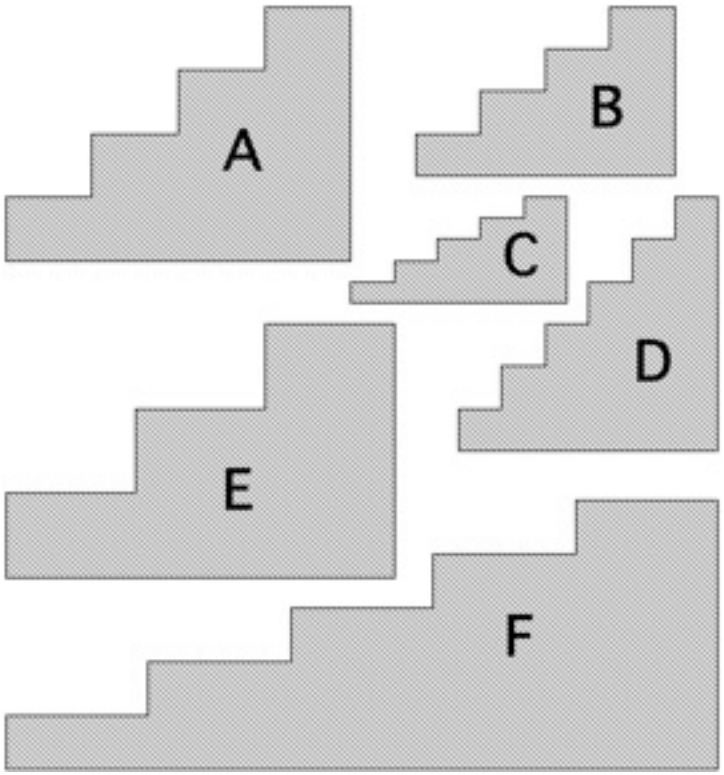
“Height of step - length of step”

a good measure of steepness?

Why is...

“Height of step ÷ length of step”

a better measure?



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Multiple representations: Percent changes

A sheet of questions *Percent changes* is given to students for homework before the lesson, or in the previous lesson.

In a sale, all prices in a shop were decreased by 20%.

After the sale they were all increased by 25%.

What was the overall effect on the shop prices?

Explain how you know.

A common misconception here is:

Price - 20% + 25% = Price +5%

Overall change is therefore an increase of 5%.

Real understanding involves knowing that we are combining multipliers:

Price x 0.8 x 1.25 = Price x 1,

There is no overall change in prices.

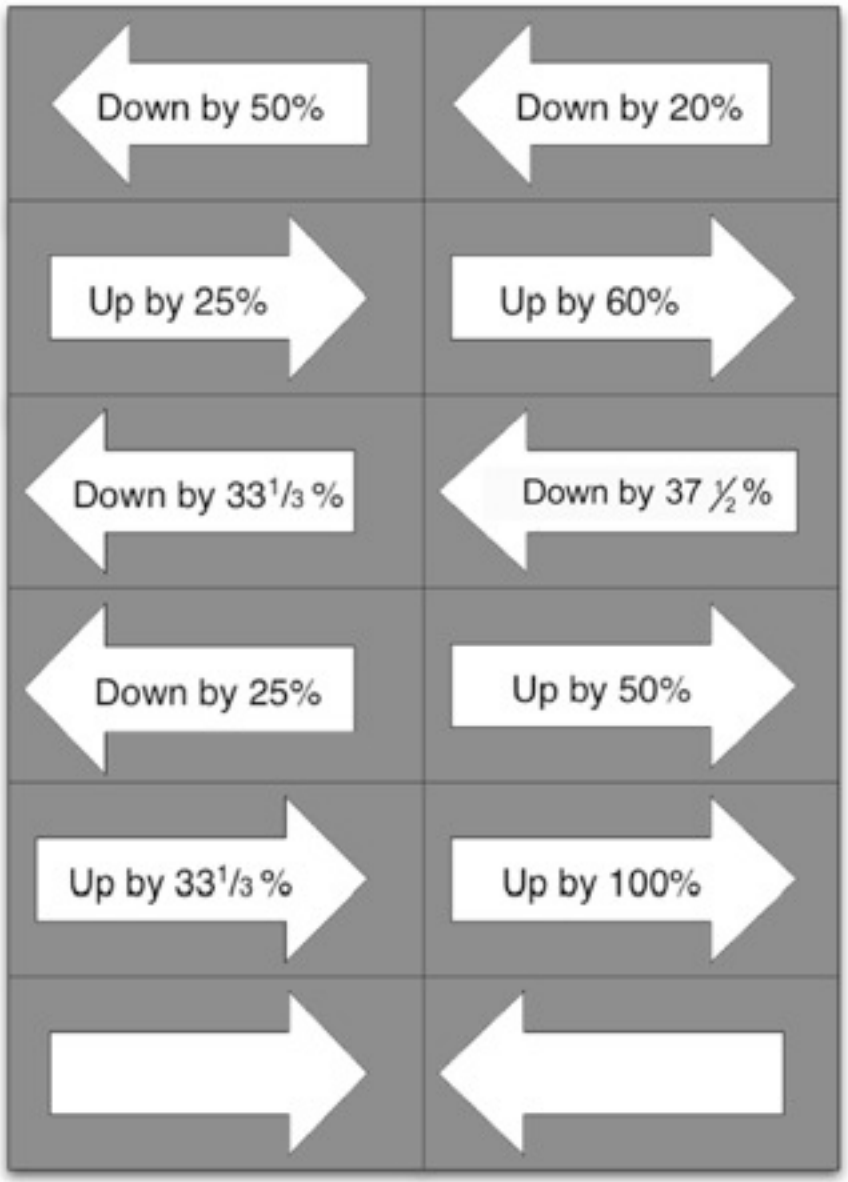
Collaborative activity

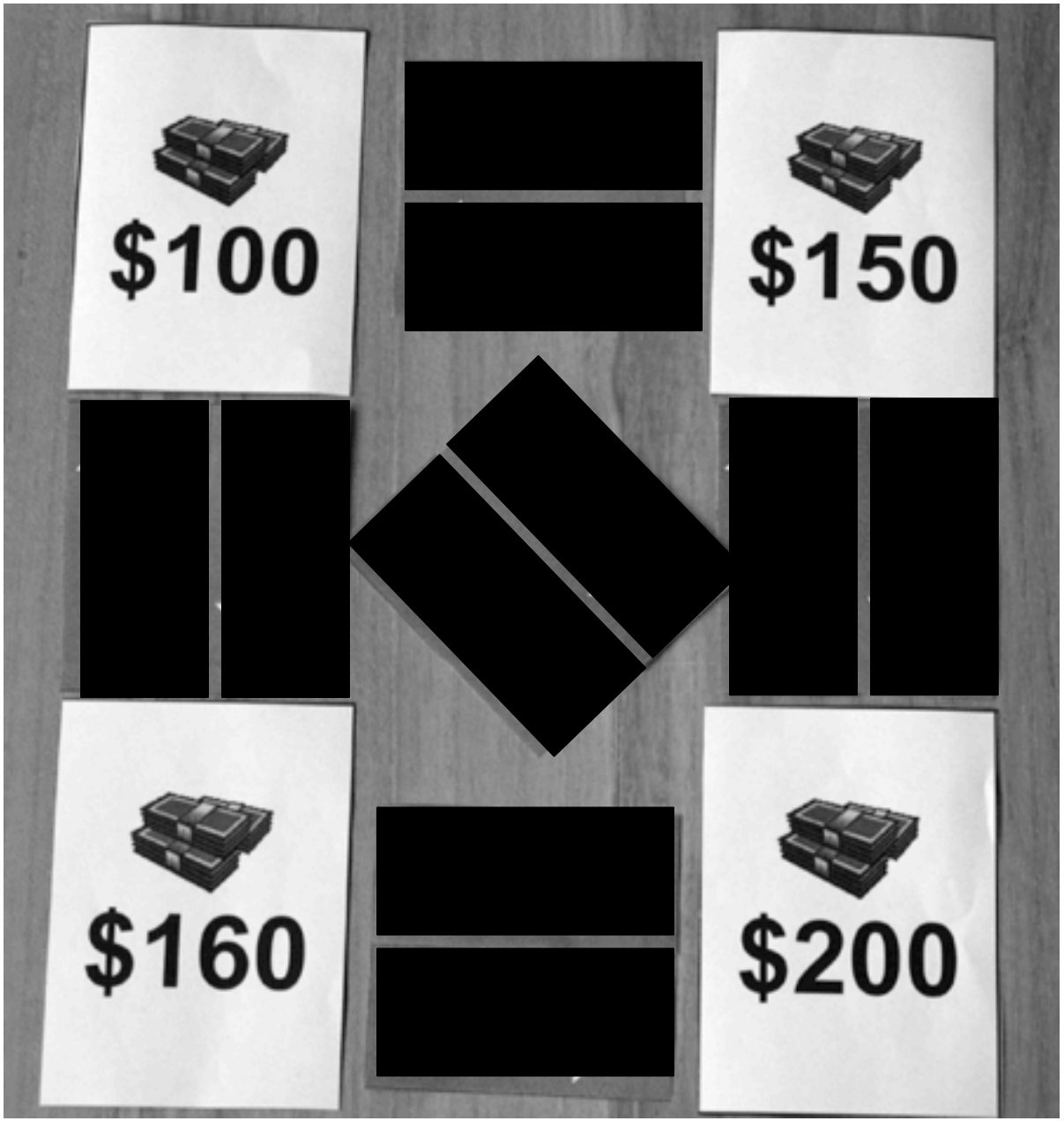
"Today, I want you to work in groups of two or three. I will give each group a set of cards."

"There is a lot of work to do today, and it doesn't matter if you don't all finish. The important thing is to learn something new, so take your time."

"I want you to work as a team. Take it in turns to place the cards on the table and explain all your reasoning to your partner."

$100
$150
$200
$160


Down by 50%
Down by 20%
Up by 25%
Up by 60%
Down by 33 1/3 %
Down by 37 1/2 %
Down by 25%
Up by 50%
Up by 33 1/3 %
Up by 100%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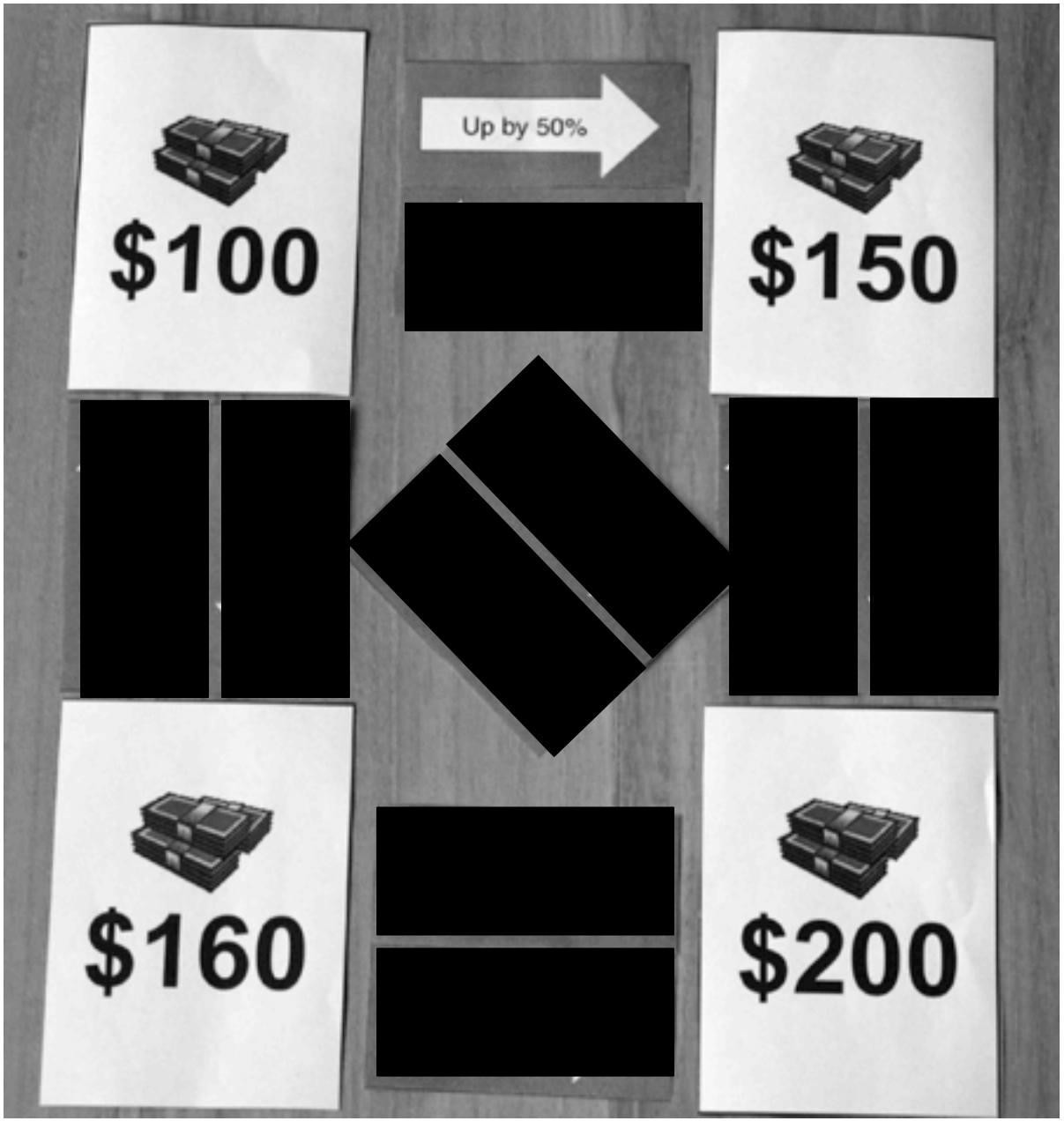
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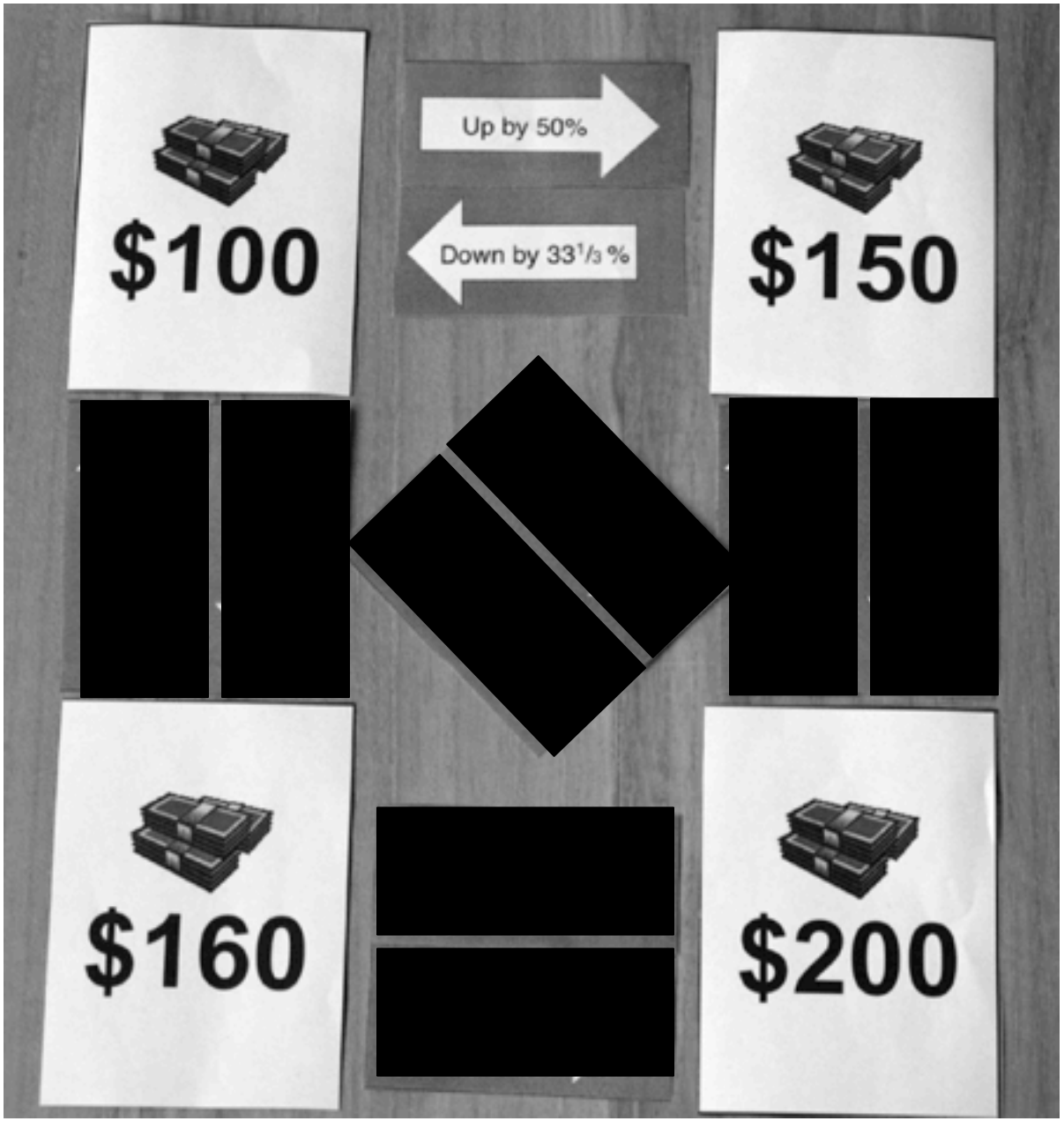
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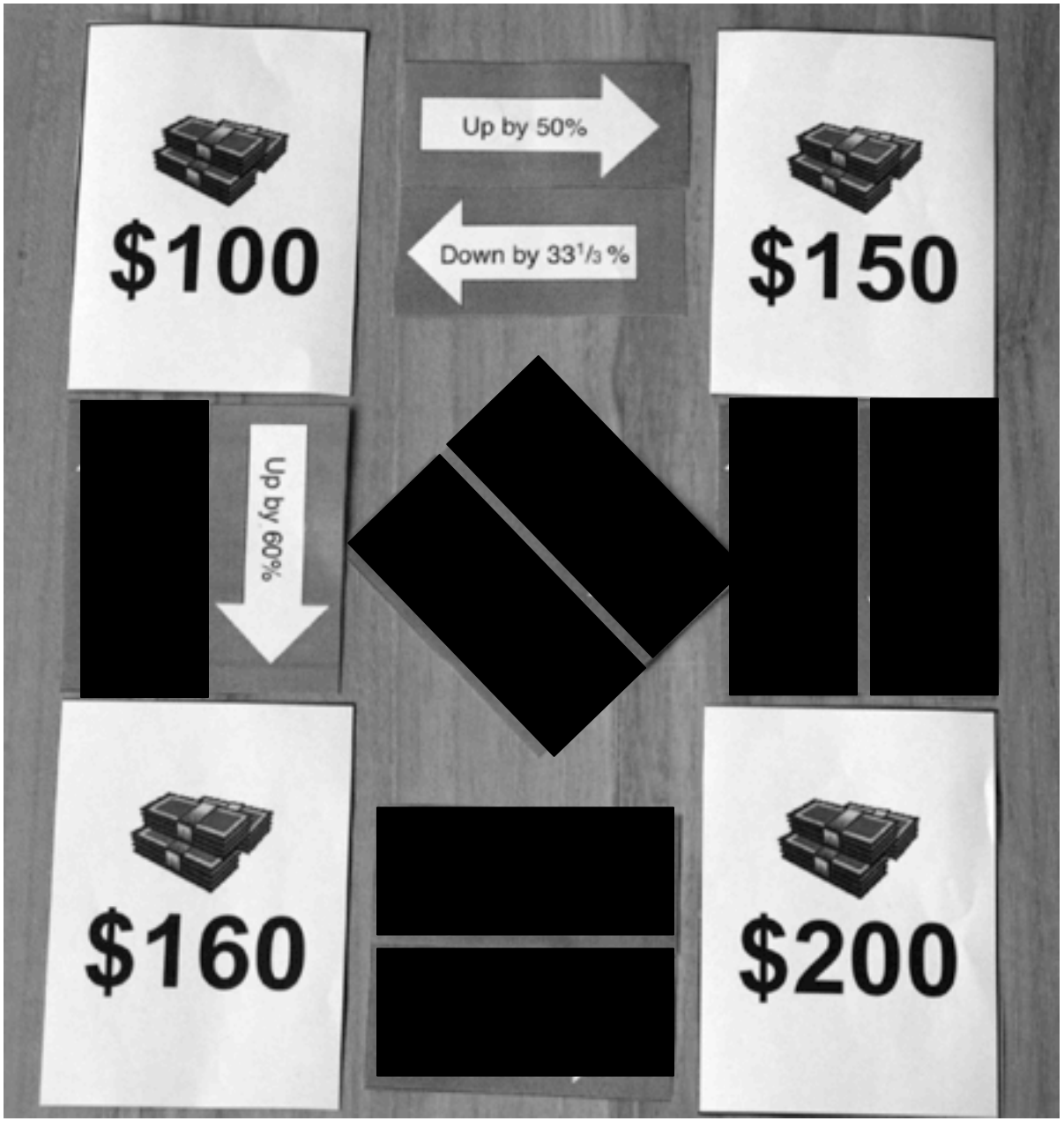
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Up by 60%
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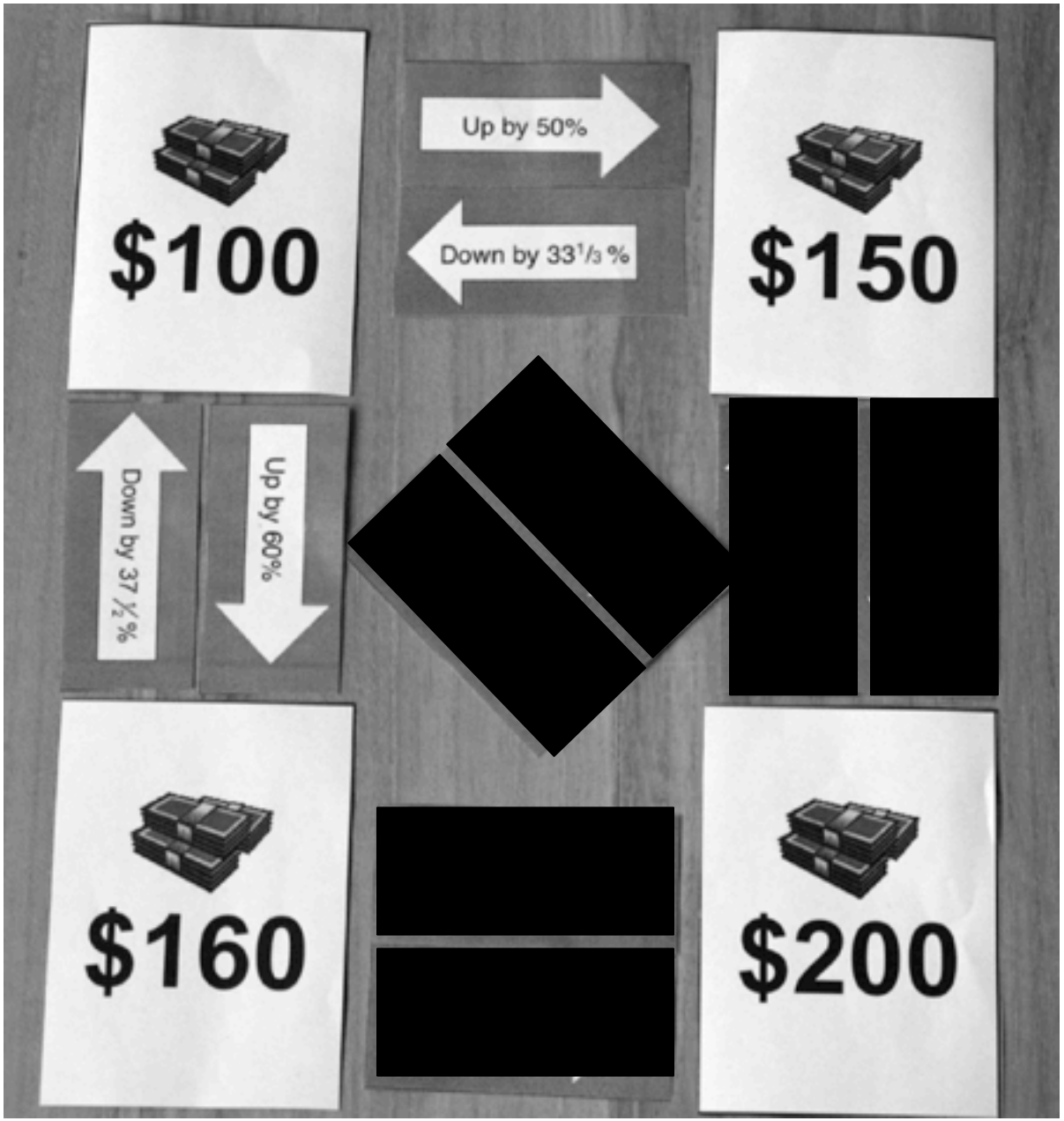
$200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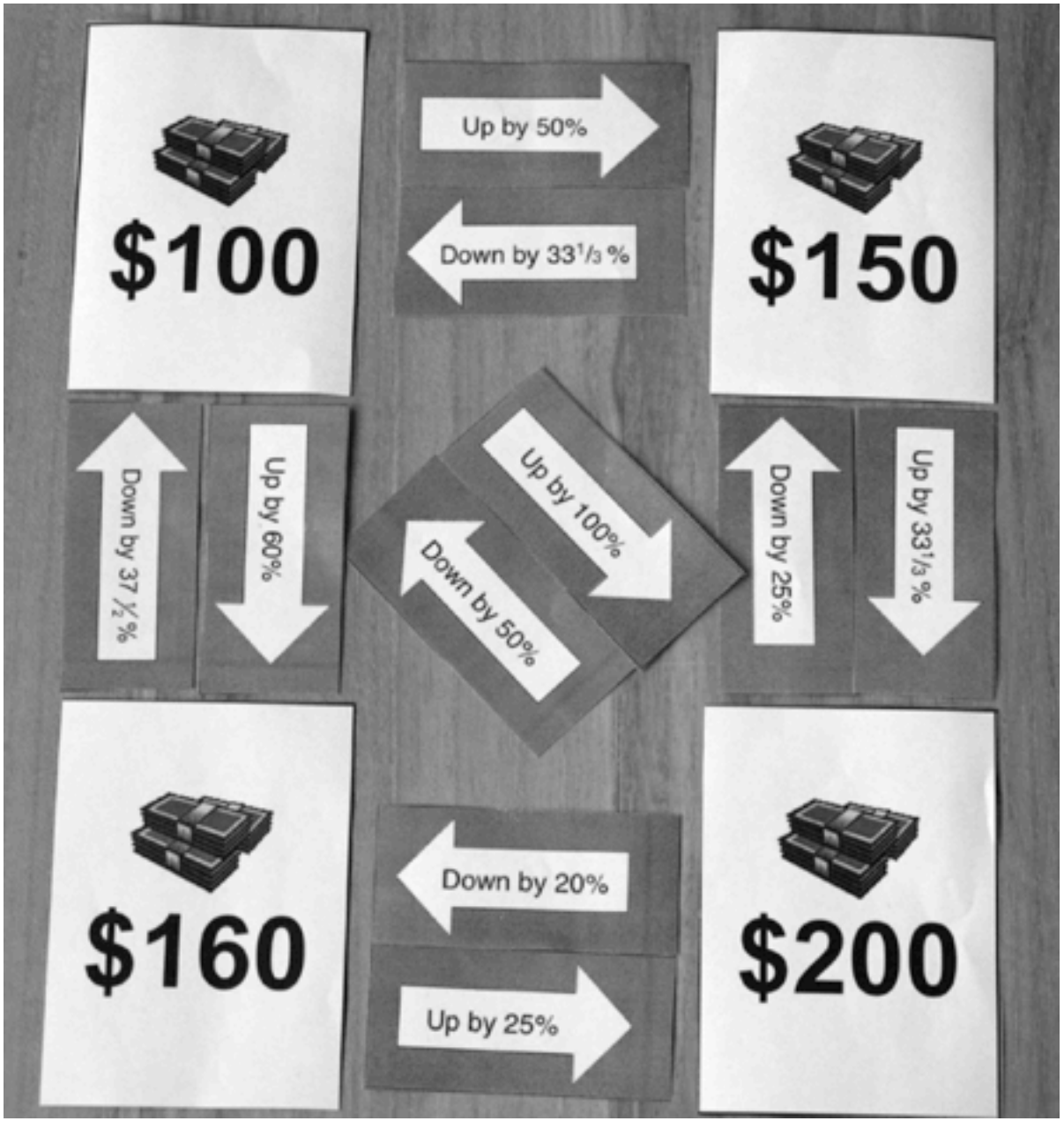
Down by 37 ½ %
Up by 60%
$160
$200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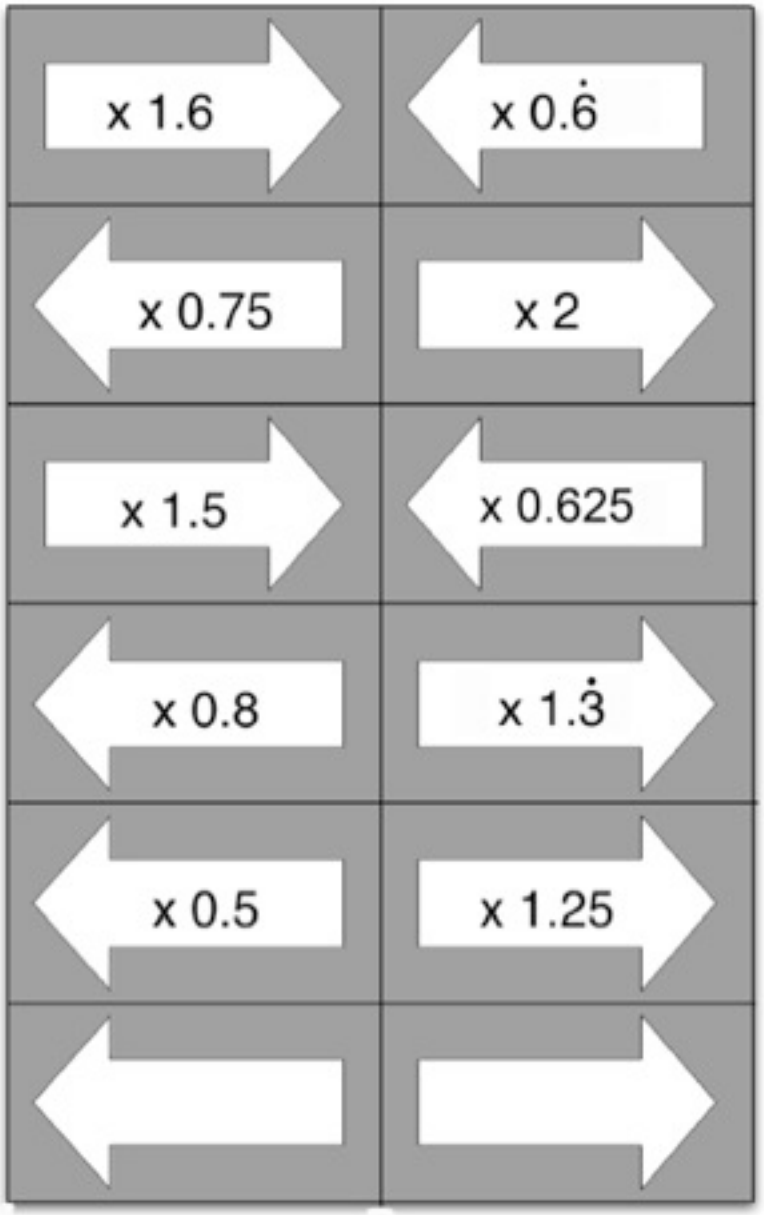
Down by 20%
Up by 25%
$200


x 1.6
x 0.6̇
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x 1.5
x 0.625
x 0.8
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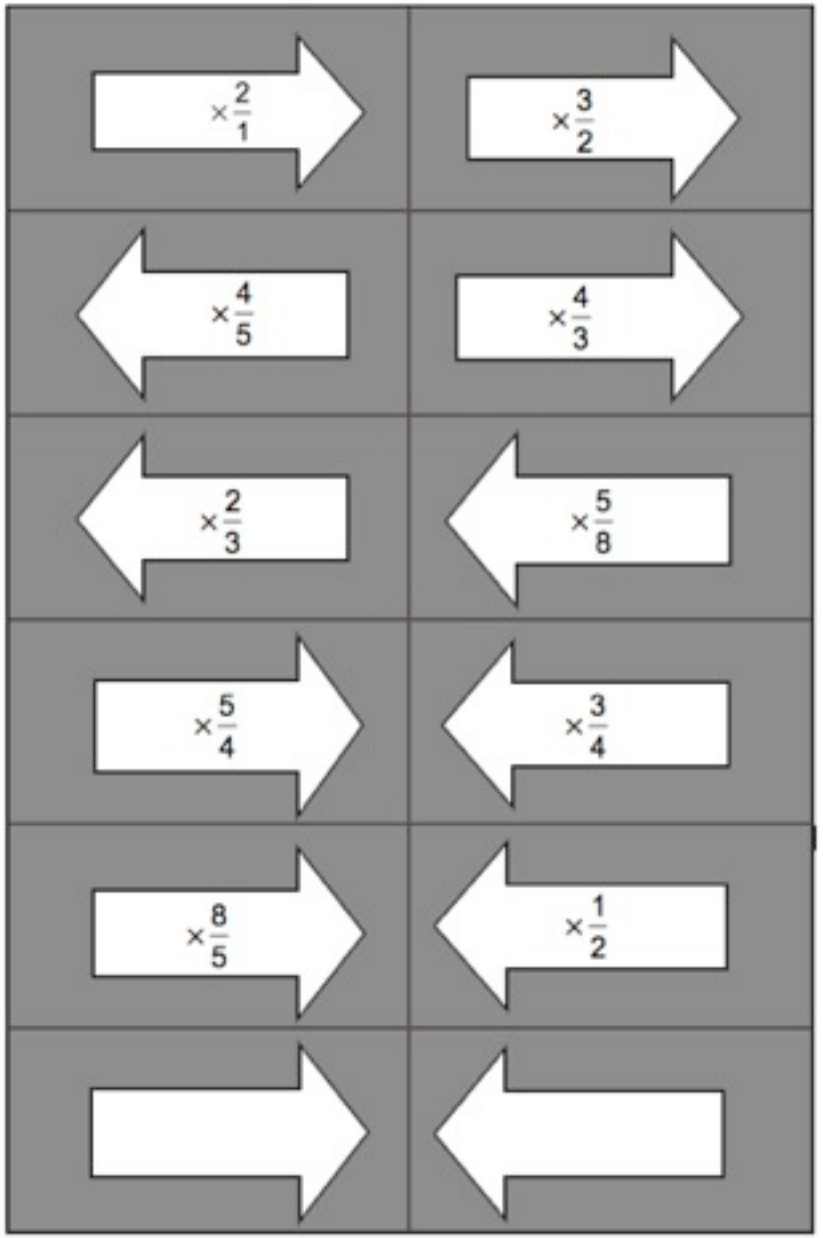
x 0.5
x 1.25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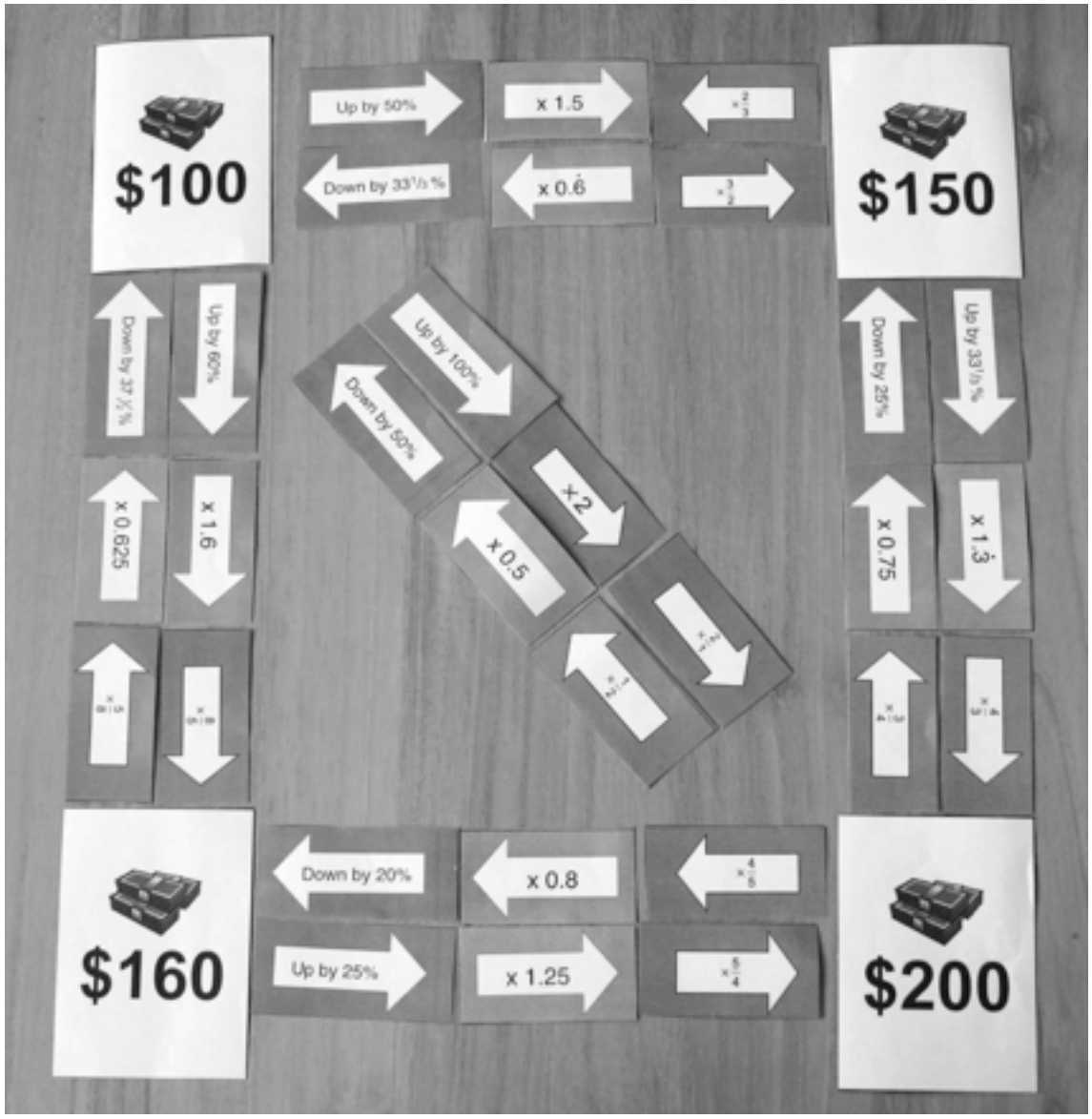
$\times\frac{1}{2}$


$100
Up by 50%
x 1.5
Down by 33⅓ %
x 0.6
$150
Down by 37½%
Up by 60%
Up by 100%
Down by 50%
x 2
x 0.5
Down by 25%
Up by 33⅓ %
x 0.625
x 1.6
x 0.75
x 1.3
$160
Down by 20%
x 0.8
Up by 25%
x 1.25
$200

Plenary discussion

Conclude the lesson by discussing and generalising what has been learned.

The generalisation can be done by first extending what has been learned to new examples:

If prices increase by 10%...

How can I say that as a decimal multiplication?

How can I write that as a fraction multiplication?

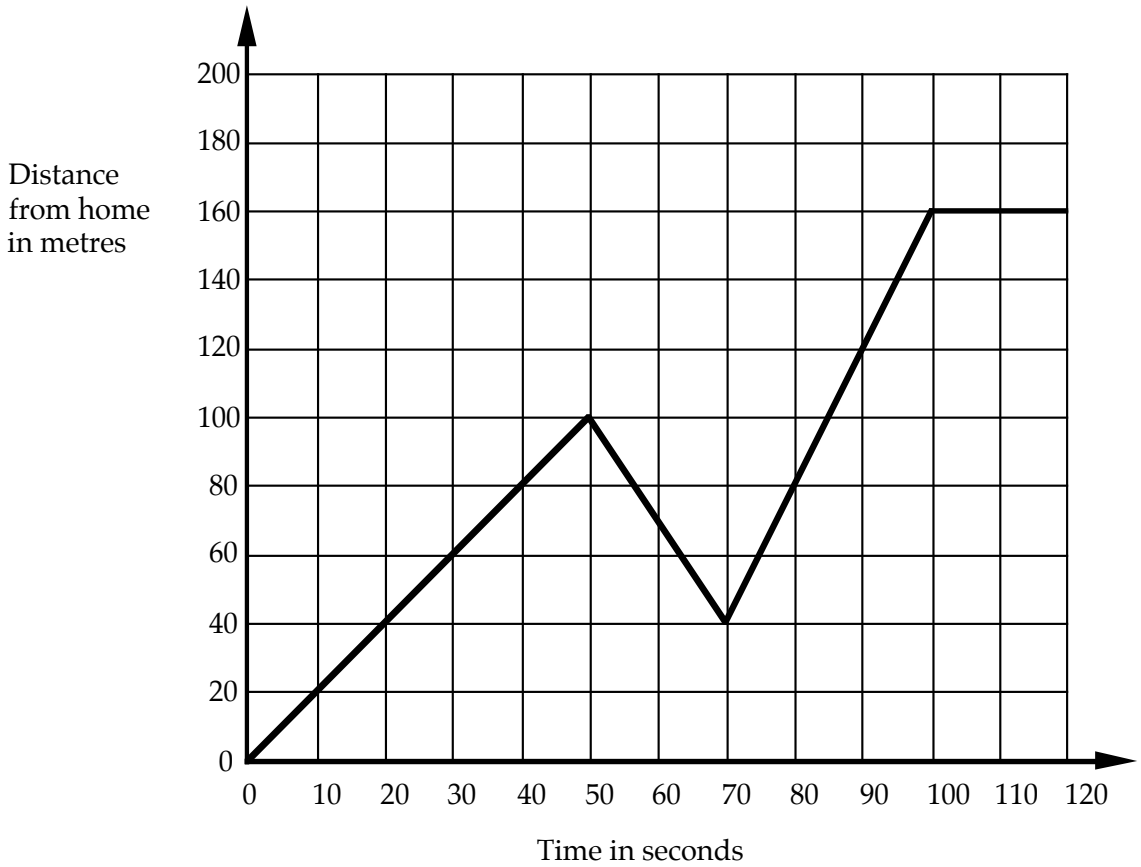
How much will prices need to go down to get back to the original price?

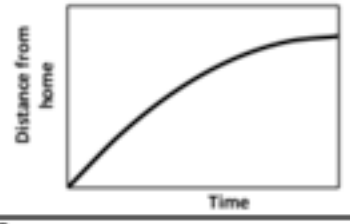
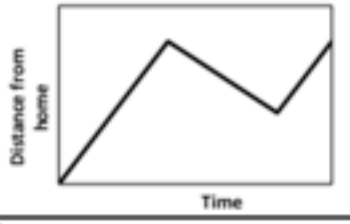
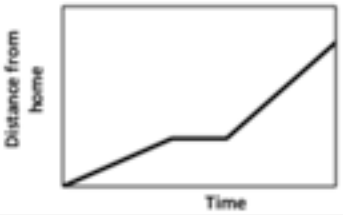
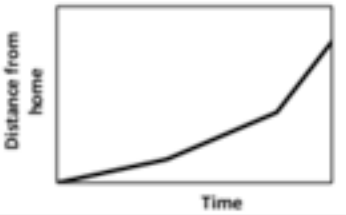
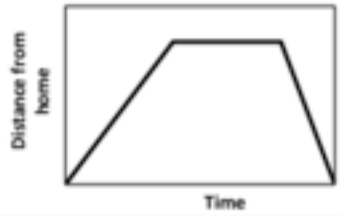
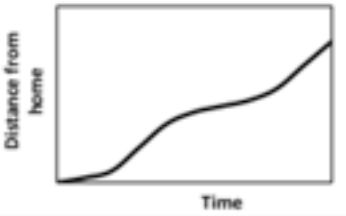
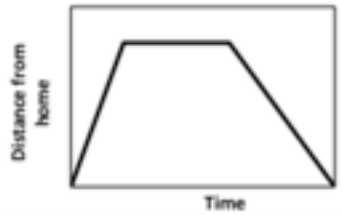
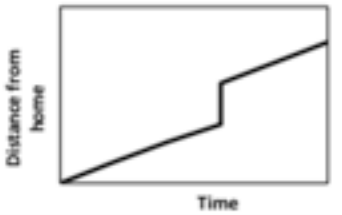
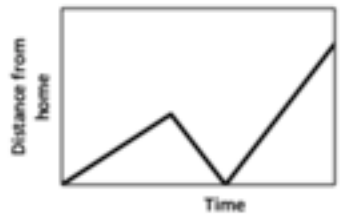
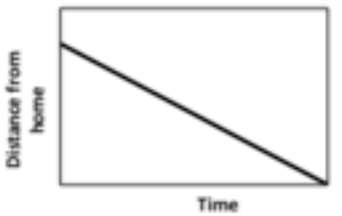
How can you write that as a decimal multiplication?

How can you write that as a fraction multiplication?

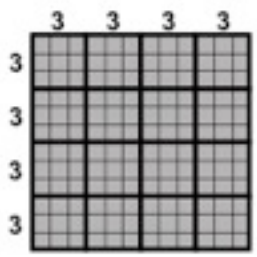
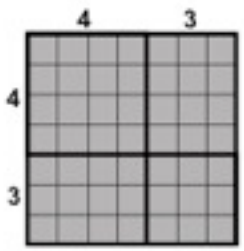
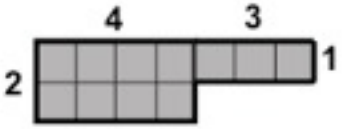
Interpreting Graphs

Every morning Jane walks along a straight road to a bus stop 160 metres from her home, where she catches a bus to college.
The graph shows her journey on one particular day. Describe what may have happened.



A.  	B. 
C.  	D. 
E.  	F. 
G.  	H. 
I.  	J. 

A. Tom ran from his home to the bus stop and waited. He realised that he had missed the bus so he walked home.	B. Opposite Tom's home is a hill. Tom climbed slowly up the hill, walked across the top and then ran quickly down the other side.
C. Tom skateboarded from his house, gradually building up speed. He slowed down to avoid some rough ground, but then speeded up again.	D. Tom walked slowly along the road, stopped to look at his watch, realised he was late, then started running.
E. Tom left his home for a run, but he was unfit and gradually came to a stop!	F. Tom walked to the store at the end of his street, bought a newspaper, then ran all the way back.
G. Tom went out for a walk with some friends when he suddenly realised he had left his wallet behind. He ran home to get it and then had to run to catch up with the others.	H. This graph is just plain wrong. How can Tom be in two places at once?
I. After the party, Tom walked slowly all the way home.	J. Make up your own story!

A 	B 
C	D 
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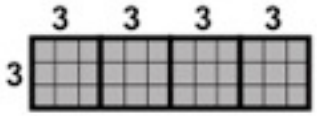
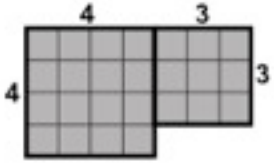
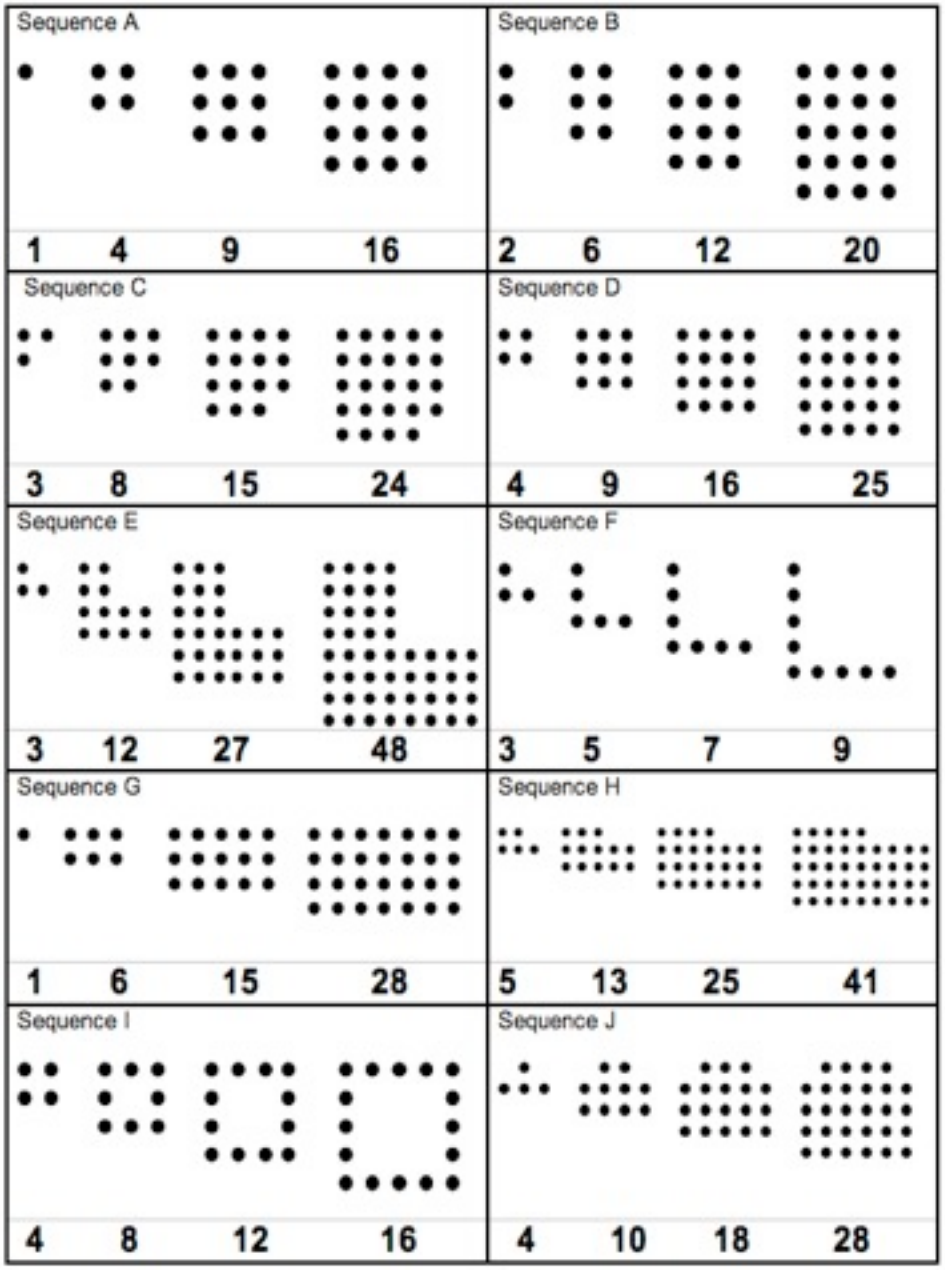
G 	H 

A $2\times(4+3)$	B $2\times4+3$
C $\frac{1}{2}(4+3)$	D $\frac{4+3}{2}$
E $2\times4+2\times3$	F 4×3^2
G $(4\times3)^2$	H $(4+3)^2$
I 4^2+3^2	J $4^2+2\times3\times4+3^2$
K $\frac{4}{2}+\frac{3}{2}$	L $\frac{1}{2}\times4+3$
M $4^2\times3^2$	N $3+4\times2$



n^2	$4n$
n^2+2n+1	n^2+2n
n^2+n	$2n+1$
$2n^2-n$	$3n^2$
n^2+3n	$(n+1)^2$
$n(n+1)$	$n(2n-1)$
$(n+1)^2-1$	$(n+1)^2-n^2$
$(2n)^2-n^2$	$(n+1)^2-(n-1)^2$
$n+n(n+2)$	$n^2+(n+1)^2$
$(n+1)(2n+1)-n$	$n+n(n-1)$

Task "genres" that generate discussion

- **Classifying, naming and defining objects**
  - what is the same and what is different?
- **Interpreting multiple representations**
  - what is another way of showing this?
- **Analysing and testing generalisations**
  - "always, sometimes or never true?"
- **Modifying problems & exploring structure**
  - what happens if I change this?
  - How will it affect this?

Testing assertions & "misconceptions"

Shopping offers "Three for the price of two" is a better offer than "30% off".	**Diagonals** The diagonals of a quadrilateral divide the quadrilateral into 4 equal areas.
Algebraic statement $\frac{a+b}{2} \geq \sqrt{ab}$	**Bigger fractions** If you add the same number to the top and bottom of a fraction, the fraction increases in value.

Always, sometimes or never true?

Always, sometimes or never true?

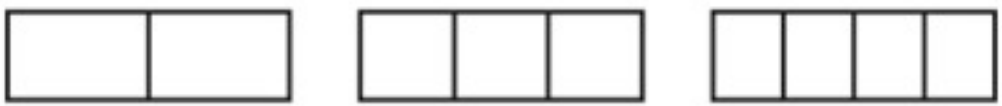
When you add the same number to the top and bottom of a fraction, the fraction increases in value.

Always, sometimes or never true?

When you add the same number to the top and bottom of a fraction, the fraction increases in value.

Draw diagrams to show:

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$



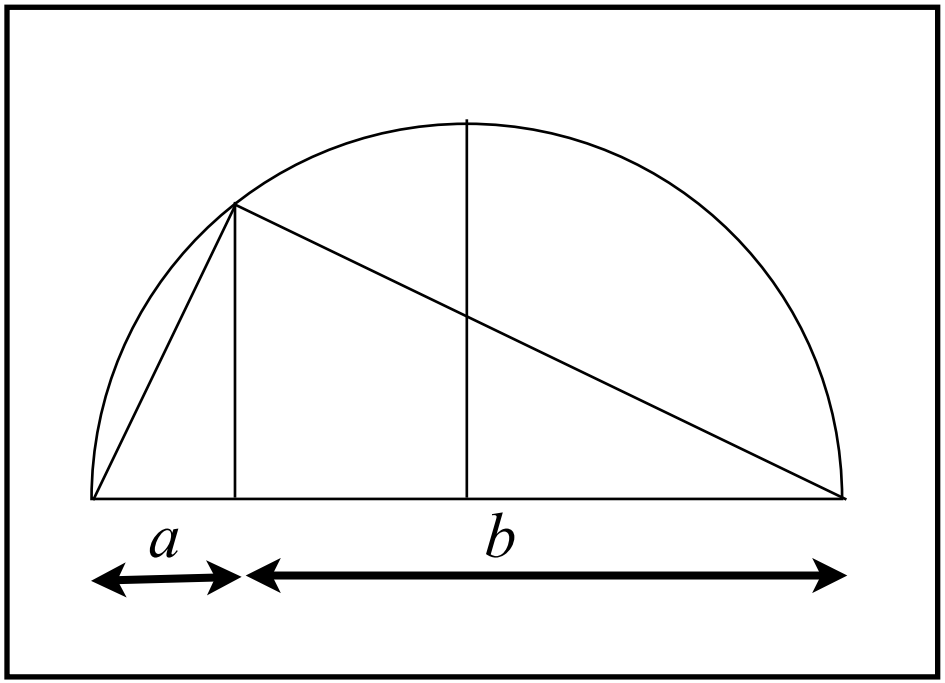
What happens to the size?
Now you try a different sequence.

Always, sometimes or never true?

$$\frac{a+b}{2} \geq \sqrt{ab}$$

Always, sometimes or never true?

$$\frac{a+b}{2} \geq \sqrt{ab}$$



Task "genres" that generate discussion

- **Classifying objects & challenging definitions**
  - what is the same and what is different?
- **Interpreting multiple representations**
  - what is another way of showing this?
- **Analysing and testing generalisations**
  - "always, sometimes or never true?"
- **Modifying problems & exploring structure**
  - what happens if I change this?
  - How will it affect this?

Making and selling candles

The cost of buying the kit (includes moulds, wax, wicks)	£ *k* = 50	
The number of candles that can be made with the kit	*n* = 60	candles
The price at which he sells each candle	£ *s* = 4	
Total profit made if all candles are sold.	£ *p* =	

Making and selling candles

	k	
The cost of buying the kit (includes moulds, wax, wicks)	£ 50	
	n	
The number of candles that can be made with the kit	60	candles
	s	
The price at which he sells each candle	£ 4	
	p	
Total profit made if all candles are sold.	£	

$$p = 60 \times 4 - 50$$

Making and selling candles

The cost of buying the kit (includes moulds, wax, wicks)	£ *k* = 50	
The number of candles that can be made with the kit	*n* = 60	candles
The price at which he sells each candle	£ *s* = 4	
Total profit made if all candles are sold.	£ *p* =	

Making and selling candles

	k	
The cost of buying the kit (includes moulds, wax, wicks)	£ 50	
	n	
The number of candles that can be made with the kit	60	candles
	s	
The price at which he sells each candle	£ 4	
	p	
Total profit made if all candles are sold.	£	

$$p = ns - k$$

Making and selling candles

		k	
The cost of buying the kit (includes moulds, wax, wicks)	£	50	
		n	
The number of candles that can be made with the kit		60	candles
		s	
The price at which he sells each candle	£		
		p	
Total profit made if all candles are sold.	£	190	

Making and selling candles

	k	
The cost of buying the kit (includes moulds, wax, wicks)	£ 50	
	n	
The number of candles that can be made with the kit	60	candles
	s	
The price at which he sells each candle	£	
	p	
Total profit made if all candles are sold.	£ 190	

$$s = \frac{190 + 50}{60}$$

Making and selling candles

		k	
The cost of buying the kit (includes moulds, wax, wicks)	£	50	
		n	
The number of candles that can be made with the kit		60	candles
		s	
The price at which he sells each candle	£		
		p	
Total profit made if all candles are sold.	£	190	

Making and selling candles

	k	
The cost of buying the kit (includes moulds, wax, wicks)	£ 50	
	n	
The number of candles that can be made with the kit	60	candles
	s	
The price at which he sells each candle	£	
	p	
Total profit made if all candles are sold.	£ 190	

$$s = \frac{p + k}{n}$$

Making and selling candles

The cost of buying the kit (includes molds, wax, wicks)	£ *k* = 50	
The number of candles that can be made with the kit	*n* =	candles
The price at which he sells each candle	£ *s* = 4	
Total profit made if all candles are sold.	£ *p* =	

Making and selling candles

The cost of buying the kit (includes molds, wax, wicks) — *k* = £ 50

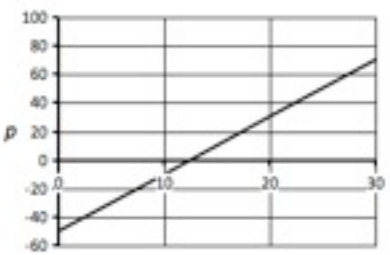
The number of candles that can be made with the kit — *n* = ____ candles

The price at which he sells each candle — *s* = £ 4

Total profit made if all candles are sold. — *p* = £ ____

$$p = 4n - 50$$

n	0	10	20	30	40	50
p	-50	-10	30	70	110	150



Making and selling candles

The cost of buying the kit (includes moulds, wax, wicks) £ ☐ *k*

The number of candles that can be made with the kit ☐ *n* candles

The price at which he sells each candle £ ☐ *s*

Total profit made if all candles are sold. £ ☐ *p*

Making and selling candles

The cost of buying the kit (includes moulds, wax, wicks) £ ___ *k*

The number of candles that can be made with the kit ___ *n* candles

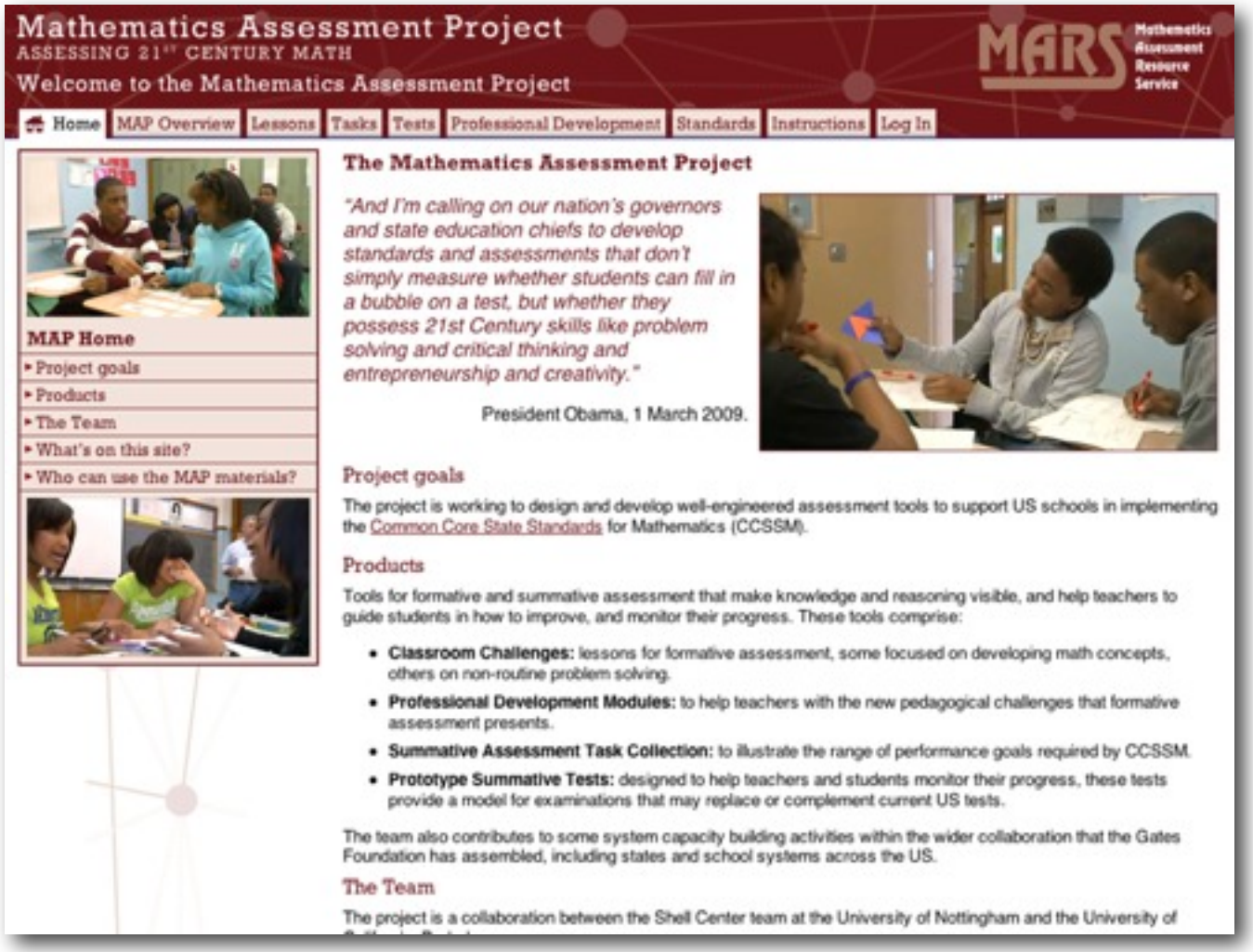
The price at which he sells each candle £ ___ *s*

Total profit made if all candles are sold. £ ___ *p*

$$p = ns - k \qquad s = \frac{p + k}{n} \qquad n = \frac{p + k}{s} \qquad k = ns - p$$

Mathematics Assessment Project (MAP)

http://map.mathshell.org/materials/



Promoting Inquiry in Maths and Science (PRIMAS)

http://primas.mathshell.org.uk/pd.htm

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For Inquiry-based, Collaborative Learning

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3. ► Learning Concepts Through IBL
4. ► Asking Questions that Promote Reasoning
5. ► Students Working Collaboratively

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About These Modules

These modules explore some of the pedagogical challenges that arise when introducing investigative, non-routine problem solving activities to the classroom.

The modules are activity-based; built around a collection of example classroom activities. The intention is that, as part of the Professional Development (PD) process, teachers will plan inquiry-based lessons to use with their own class and, at a later meeting, report back on their experiences.

Each module includes a PD session guide and handouts for teachers, as well as sample classroom materials and suggested lesson plans. Several of the lessons include the use of simple computer software.

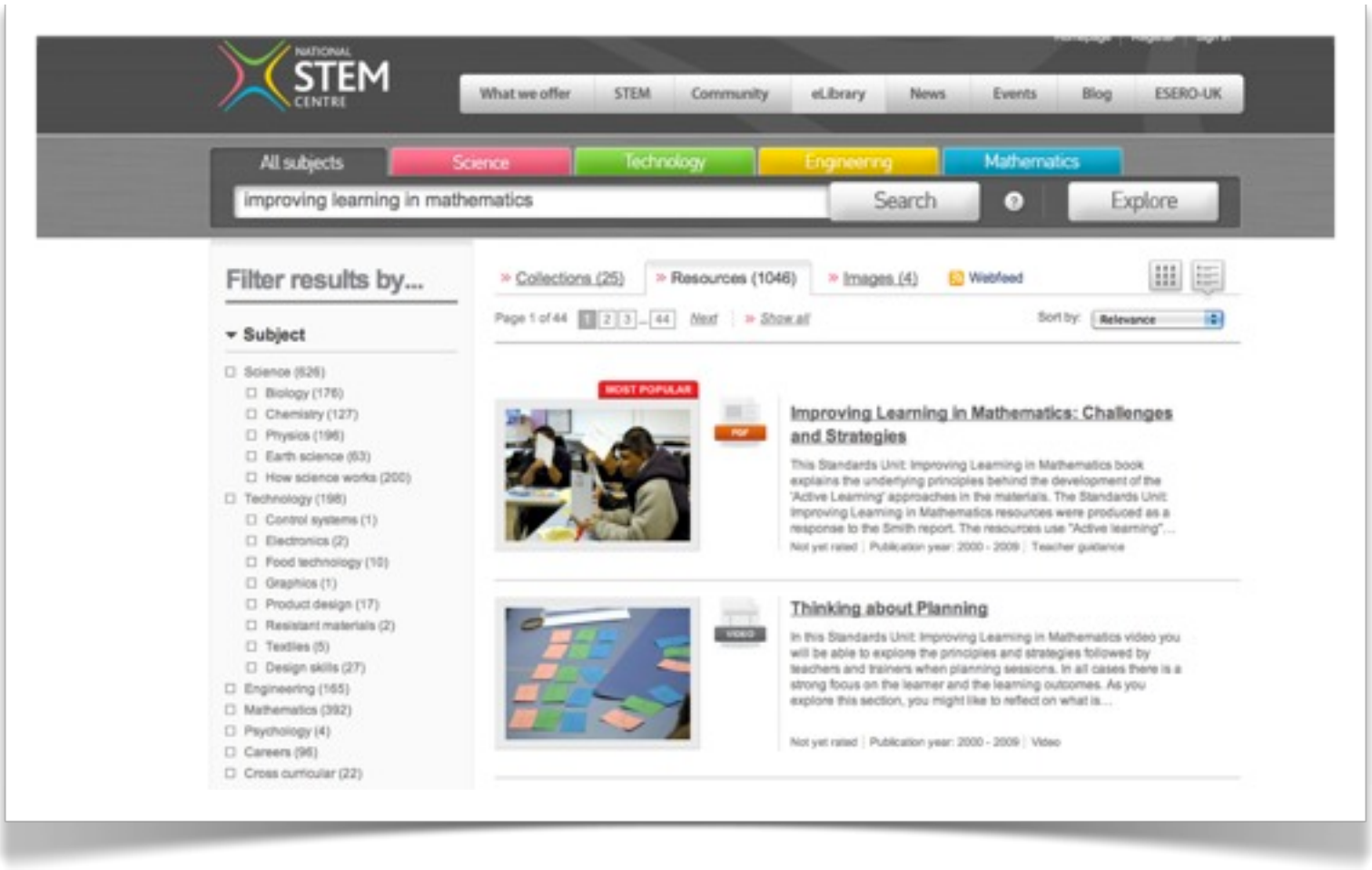
Also included are several video sequences showing teachers trying these materials with their own classes. These form a stimulating basis for discussion during the PD sessions.

The modules have been compiled for PRIMAS from professional development materials developed by the Shell Centre team at the Centre for Research in Mathematics Education, University of Nottingham. These materials were originally written for two projects:

- Bowland Maths is a project aimed at developing the key processes involved in mathematical problem solving, using a range of realistic and fantasy scenarios. This was funded by the Bowland Charitable Trust.
- Improving Learning in Mathematics is a project aimed at developing the understanding of mathematical concepts. This was originally funded by the Department for Education and Skills Standards Unit.

Bowland Maths

http://www.bowlandmaths.org.uk/

Improving Learning in Mathematics

http://www.nationalstemcentre.org.uk